Implicit-Explicit Multistep Methods for Fast-Wave Slow-Wave Problems

Dale Durran   Peter Blossey
University of Washington

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Splitting Up the PDE

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{f}(\mathbf{u}) + \mathbf{L}\mathbf{u},$$

where

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- $\mathbf{L}$ is the matrix associated with a linear operator modeling processes with short timescales
- $\mathbf{f}(\mathbf{u})$ is everything else

IMEX Multistep Approximation

$$\sum_{k=-M}^{1} \alpha_k \mathbf{q}^{n+k} = \Delta t \left[\sum_{k=-M}^{0} \beta_k \mathbf{f}(\mathbf{q}^{n+k}) + \sum_{k=-M}^{1} \nu_k \mathbf{L} \mathbf{q}^{n+k} \right].$$

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- (α_k, β_k) define the explicit method
- (α_k, ν_k) define the implicit scheme

Common in Atmospheric Models – T2θ/LF

Implicit: Trapezoidal over $2\Delta t$

$$\frac{\mathbf{q}^{n+1} - \mathbf{q}^{n-1}}{2\Delta t} = \theta \mathbf{L}\mathbf{q}^{n+1} + (1-\theta)\mathbf{L}\mathbf{q}^{n-1}$$

- Damping if $\theta > 0.5$

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Explicit: Asselin-filtered leapfrog

$$\begin{aligned} \frac{\mathbf{q}^{n+1} - \tilde{\mathbf{q}}^{n-1}}{2\Delta t} &= \mathbf{f}(\mathbf{q}^n) \\ \tilde{\mathbf{q}}^n &= \mathbf{q}^n + \gamma\left(\tilde{\mathbf{q}}^{n-1} - 2\mathbf{q}^n + \mathbf{q}^{n+1}\right), \end{aligned}$$

- Typically $0.05 \leq \gamma \leq 0.2$

Weaknesses

Asselin-filtered leapfrog:

- first-order
- computational mode is only weakly damped
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Off-centered trapezoidal method:

- first-order
- damping is not very scale selective

Alternate backward scheme – BDF2/BX22

Implicit:

$$\frac{\frac{3}{2}\mathbf{q}^{n+1} - 2\mathbf{q}^{n} + \frac{1}{2}\mathbf{q}^{n-1}}{\Delta t} = \mathbf{L}\mathbf{q}^{n+1}$$

- *L*-stable, 2nd order
- Difficult to couple with an explicit time difference

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Explicit:

$$\frac{\frac{3}{2}\mathbf{q}^{n+1} - 2\mathbf{q}^{n} + \frac{1}{2}\mathbf{q}^{n-1}}{\Delta t} = 2\mathbf{f}(\mathbf{q}^{n}) - \mathbf{f}\left(\mathbf{q}^{n-1}\right)$$

- 2nd order
- Amplifies oscillatory solutions

Can we do better?

Try Adam's methods

A-stable Implicit Adams methods

Trapezoidal

$$\frac{\mathbf{q}^{n+1} - \mathbf{q}^n}{\Delta t} = \frac{1}{2}\left(\mathbf{L}\mathbf{q}^{n+1} + \mathbf{L}\mathbf{q}^n\right)$$

A-stable Implicit Adams methods

Trapezoidal

$$\frac{\mathbf{q}^{n+1} - \mathbf{q}^n}{\Delta t} = \frac{1}{2}\left(\mathbf{Lq}^{n+1} + \mathbf{Lq}^n\right)$$

AM2* (Nevanlinna and Liniger, 1979)

$$\frac{\mathbf{q}^{n+1} - \mathbf{q}^n}{\Delta t} = \frac{3}{4}\mathbf{Lq}^{n+1} + \frac{1}{4}\mathbf{Lq}^{n-1}$$

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$$\frac{\mathbf{q}^{n+1}-\mathbf{q}^{n}}{\Delta t}=\frac{3}{4}\mathbf{L}\mathbf{q}^{n+1}+\frac{1}{4}\mathbf{L}\mathbf{q}^{n-1}$$

AI22 (new)

$$\frac{\mathbf{q}^{n+1}-\mathbf{q}^{n}}{\Delta t}=\frac{5}{4}\mathbf{L}\mathbf{q}^{n+1}-\mathbf{L}\mathbf{q}^{n}+\frac{3}{4}\mathbf{L}\mathbf{q}^{n-1}$$

The Explicit Step

3-step Adams-Bashforth: AB3

$$\frac{\mathbf{q}^{n+1}-\mathbf{q}^{n}}{\Delta t}=\frac{23}{12}\mathbf{f}(\mathbf{q}^{n})-\frac{4}{3}\mathbf{f}\left(\mathbf{q}^{n-1}\right)+\frac{5}{12}\mathbf{f}\left(\mathbf{q}^{n-2}\right)$$

- Optimal 3rd-order explicit Adams method
- Treats oscillatory solutions well

Oscillations Forced at Two-Frequencies

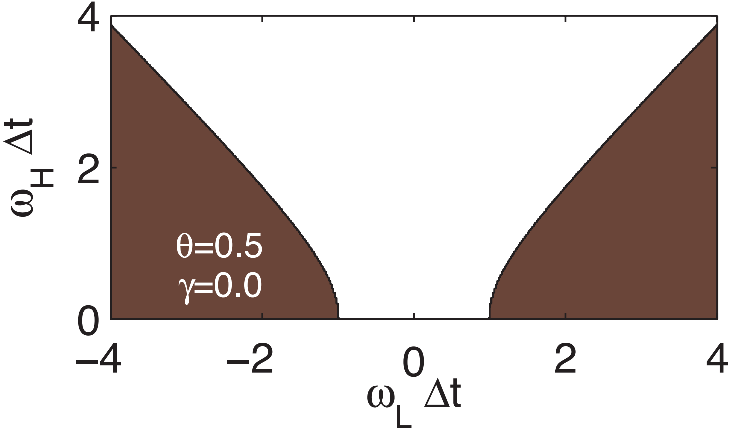
$$\frac{\partial q}{\partial t} = i\omega_L q + i\omega_H q$$

- ω_H is the high-frequency forcing.
- ω_L is the low-frequency forcing.

Leapfrog-trapezoidal amplification factor

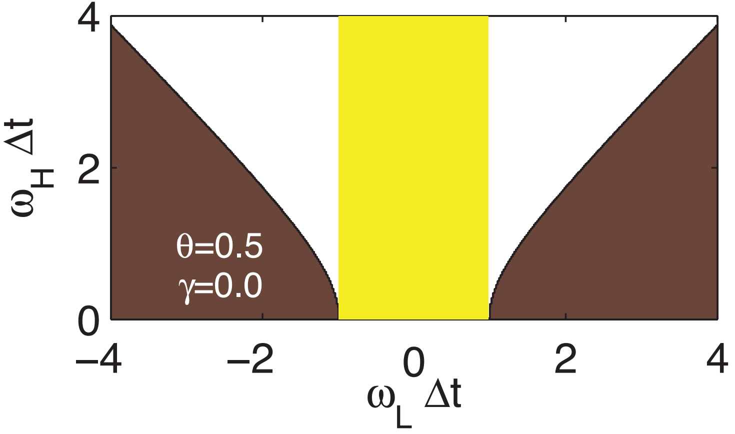
No Asselin, no off-centering

$|A| \leq 1$ throughout white region



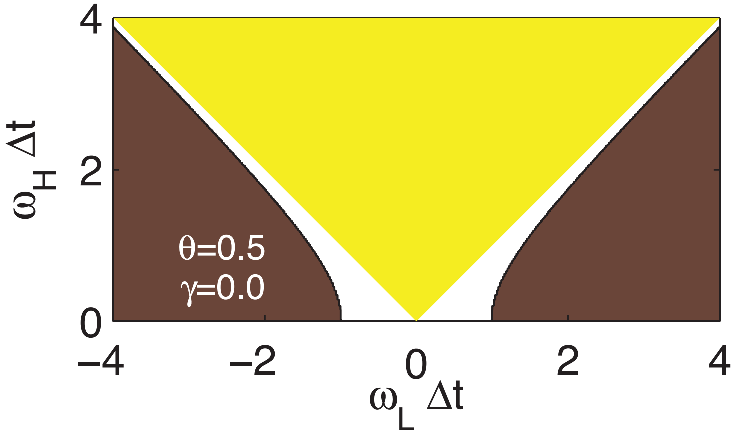
Leapfrog-trapezoidal amplification factor

Stable whenever the explicit part alone would be stable

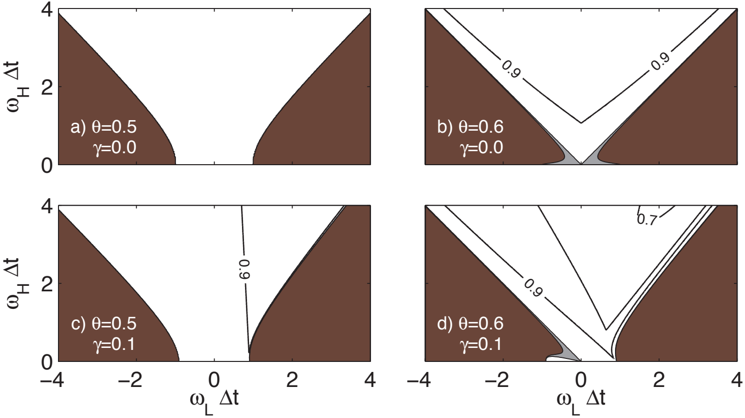


Leapfrog-trapezoidal amplification factor

Stable whenever $|\omega_\mathrm{L}| \le \xi|\omega_\mathrm{H}|$; here $\xi = 1$



Influence of Asselin filtering and off-centering



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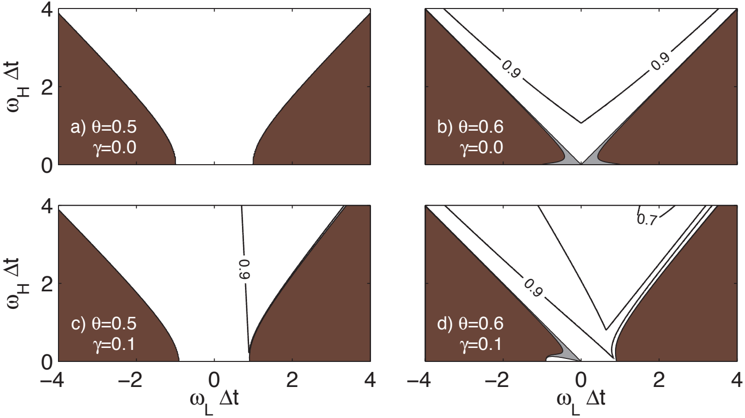
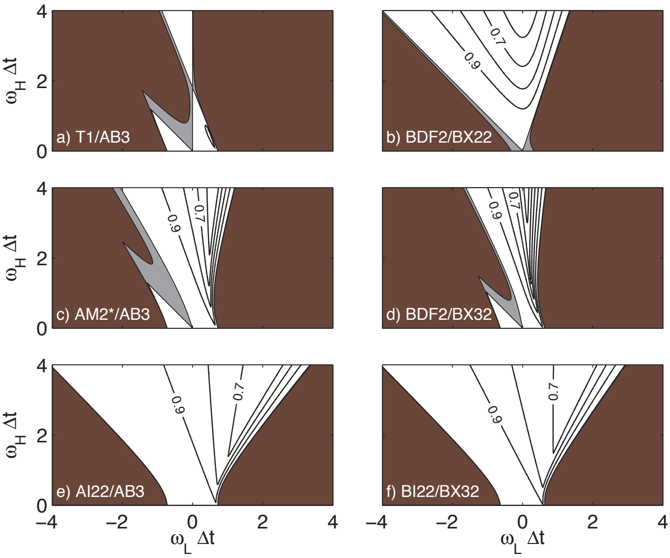
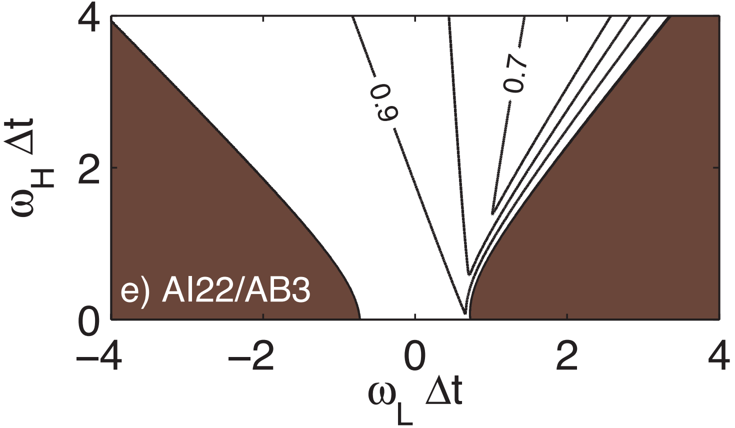


Off-centering the trapezoidal spoils stability in limit $\omega_{\mathrm{H}}\Delta t \rightarrow 0$.

Amplification factors for other schemes

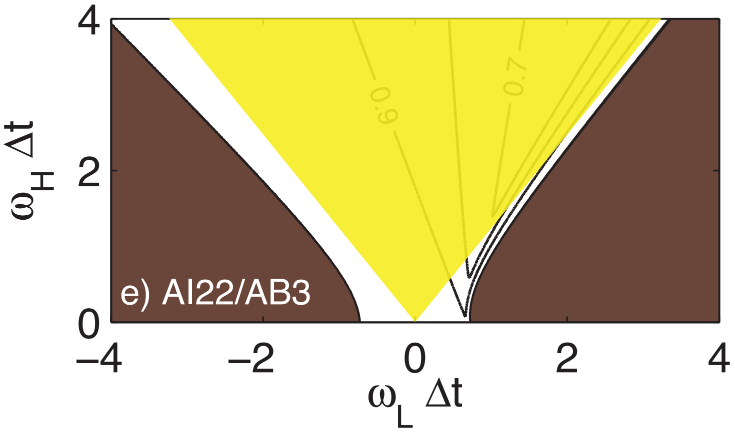


AI22/AB3 stability region



AI22/AB3 stability region

Stable whenever $|\omega_L| \le \xi|\omega_H|$; here $\xi = 1.23$



Compressible Boussinesq System

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) u + \underbrace{\frac{\partial P}{\partial x}}_{\mathrm{s}} = 0, \tag{1}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) w + \underbrace{\frac{\partial P}{\partial z}}_{\mathrm{s}} = \underbrace{b}_{\mathrm{b}}, \tag{2}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) b + \underbrace{N^2 w}_{\mathrm{b}} = 0, \tag{3}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) P + \underbrace{c_{\mathrm{s}}^2 \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)}_{\mathrm{s}} = 0. \tag{4}$$

Explicit or implicit buoyany?

Buoyancy explicit: Stability condition is

$$(|Uk_{\mathrm{max}}| + N)\Delta t < C,$$

where C is

- 1 for T2θ/LF without Asselin filtering or off centering
- 0.72 for AI22/AB3

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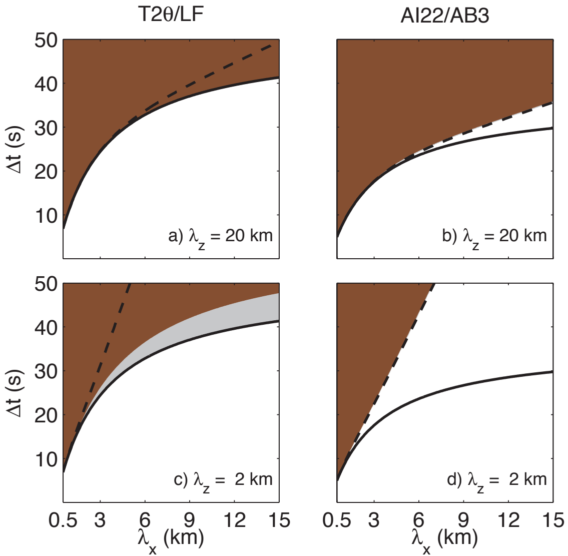
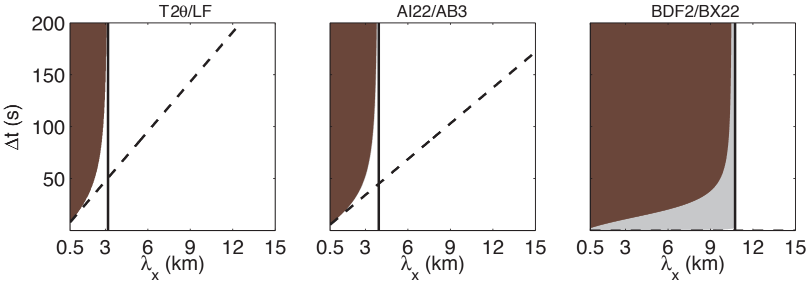
Sound waves implicit; buoyancy explicit



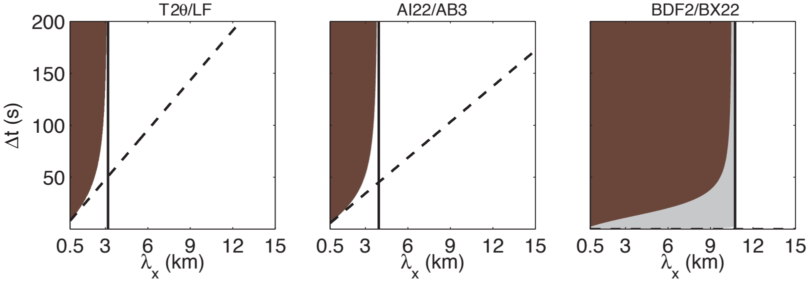
Dashed line: stability boundary for fully explicit gravity waves

Sound waves and buoyancy implicit



$$\lambda_z = 20 \text{ km}$$

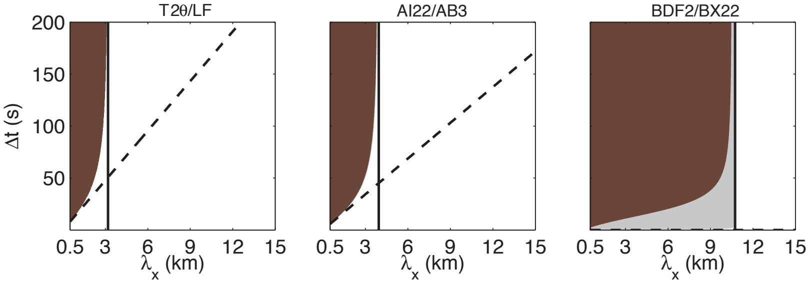
Nonhydrostatic limit: wave speed slower than U



Dashed line: $\tilde{\omega}_{\mathrm{L}} \Delta t = C$, where

$$\tilde{\omega}_{\mathrm{L}} = |Uk|.$$

Wave speed exceeds U



Solid line: $\tilde{\omega}_{\mathrm{H}}/\tilde{\omega}_{\mathrm{L}} = \xi$, where

$$\tilde{\omega}_{\mathrm{H}} = \frac{N|k|}{\left(k^2 + l^2\right)^{1/2}}; \quad \xi = \begin{cases} 1 & \text{T2}\theta\text{/LF} \\ 1.23 & \text{AI22/AB3} \\ 3 & \text{BDF2/BX22} \end{cases}$$

Numerical simulations

Fixed spatial discretization, explore convergence in time to compressible solution computed with very small Δt.

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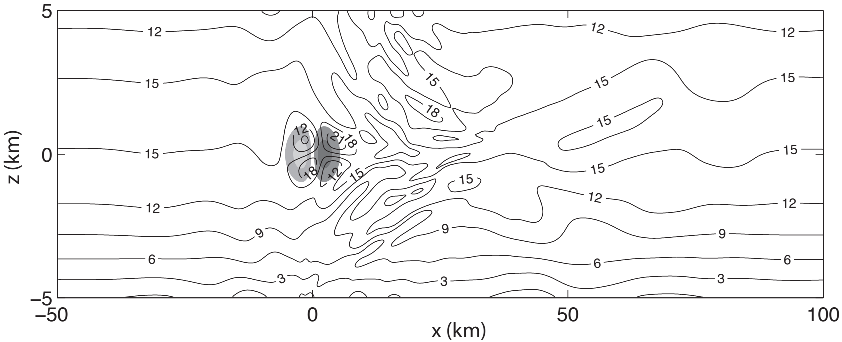
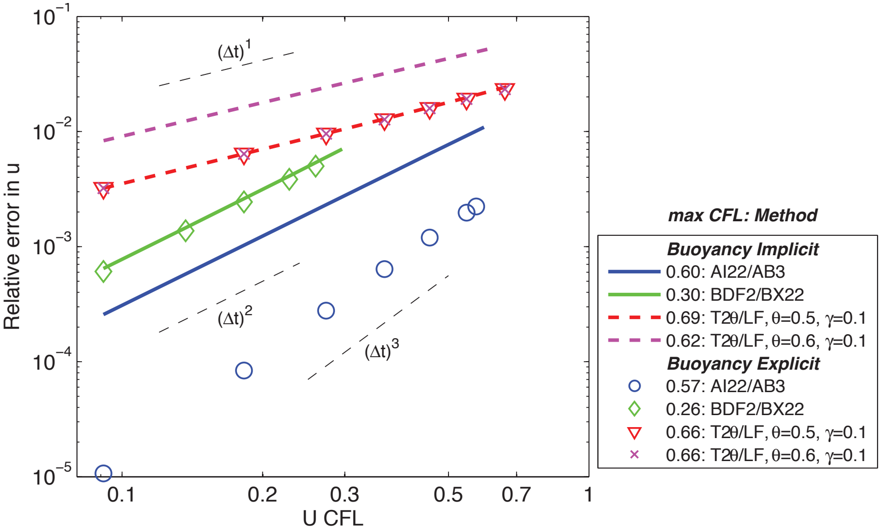
- Nonlinear compressible Boussinesq system, $c_s = 350$ m s^{-1}
- Mean shear flow, $5 \leq U(z) \leq 15$
- Constant mean static stability, $N = .01$ s^{-1}
- Periodic lateral BC, rigid top and bottom
- Buoyancy waves generated by compact nondivergent forcing

Time-converged solution

u contours at 3000 s; shading shows streamlines of forcing field

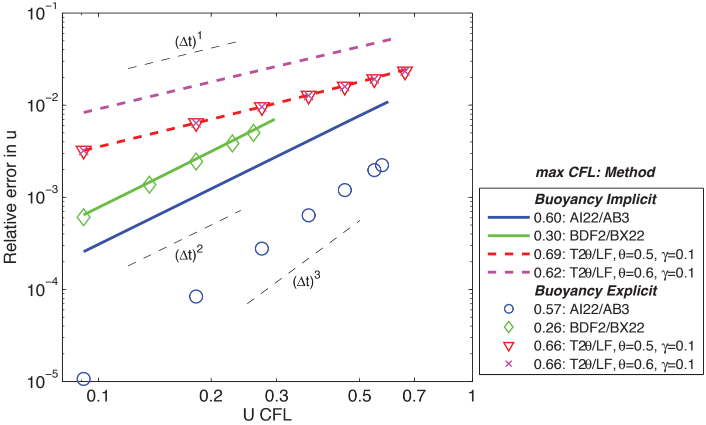


Empirical convergence rates



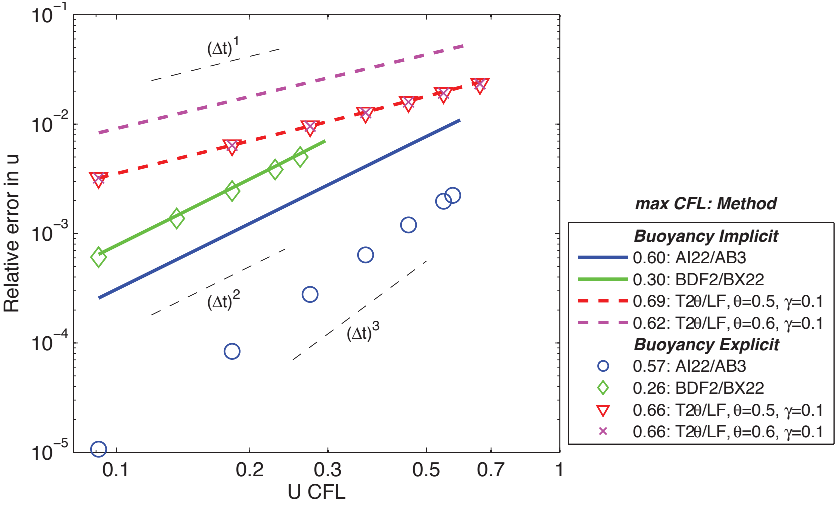
Explicit buoyancy can improve accuracy

- forward biased T2θ/LF ($\theta = 0.6$)
- AI22/AB3



Improvement at almost no CPU cost

14% reduction in maximum Δt for AI22/AB3 relative to Asselin-filtered T2θ/LF



Conclusions

New IMEX method AI22/AB3

- Higher accuracy than T2θ/LF

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- Better stability
  - Computational modes strongly damped
  - Diffusion handled better
- Minimal extra cost relative to Asselin-filtered leapfrog-trapezoidal scheme
  - 14% reduction in maximum Δt
  - One extra level of storage for explicit forcing terms

Reference

Durran, D.R., and P.N. Blossey, 2011: Implicit-Explicit Multistep Methods for Fast-Wave Slow-Wave Problems. *Mon. Wea. Rev.,* submitted.

Available from www.atmos.washington.edu/∼durrand/