# Hydrostatic Hamiltonian Particle-Mesh (HPM) methods for atmospheric modeling.

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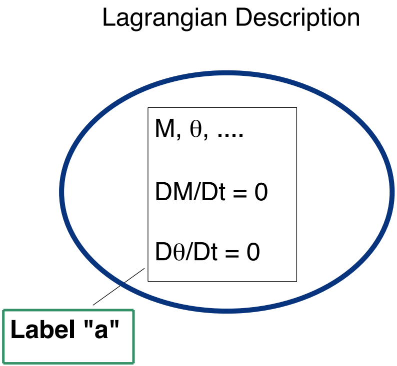
# Hamiltonian approach for geophysical flows.

- Challenge: Simultaneous fulfillment of conservation of energy (to be Hamiltonian) and robustness against unbalanced spurious high-frequency waves

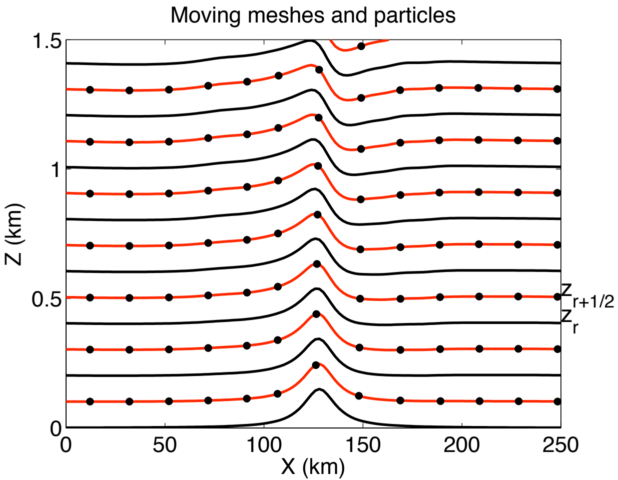
- **Hamiltonian Particle-Mesh (HPM) methods**:
  A combination of Smoothing Particle Hydrodynamics (SPH) and Particle In Cell (PIC) methods in such way that the **truncated equations are Hamiltonian + Regularization to smooth out high-frequency waves**.

# HPM: A Lagrangian framework for the description of fluid dynamics.




At the Position **q**

# Meshes and particles



# Density approximation and mass conservation

$$\rho_{i,\gamma+1/2}(t) = \frac{1}{z_{i,\gamma+1}(t) - z_{i,\gamma}(t)} \left[ \sum_{\alpha} m_{\alpha,\gamma+1/2} \hat{\psi}_i(x_{\alpha,\gamma+1/2}(t)) \right].$$

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$$\begin{aligned} &\sum_{i,\gamma} \rho_{i,\gamma+1/2}(t) \Delta x [z_{i,\gamma+1}(t) - z_{i,\gamma}(t)] \\ &= \sum_{i,\gamma} \left[ \sum_{\alpha} m_{\alpha,\gamma+1/2} \hat{\psi}_i(x_{\alpha,\gamma+1/2}(t)) \right] \Delta x \\ &= \sum_{\alpha,\gamma} m_{\alpha,\gamma+1/2} \left[ \sum_{i} \psi_i(x_{\alpha,\gamma+1/2}(t)) \right] \\ &= \sum_{\alpha,\gamma} m_{\alpha,\gamma+1/2}. \end{aligned}$$

# Discrete variational method

$$\ddot{q} = -\nabla_q V(q), \ \ J[q] = \int_{t_0}^{t_1} L[q, \dot{q}, t] dt$$

Discrete Hamiltonian principle: At a time level $n$

$$L^n(q^n) = \frac{1}{2}\left(\frac{q^n - q^{n-1}}{\Delta t}\right)^2 - V(q^n), \quad \text{where} \quad n = 1, \ldots, N.$$

$$J_{\Delta t} = \sum_{n=1}^{N} L^n \Delta t$$

$$\frac{\partial J_{\Delta t}}{\partial q^j} = -\frac{(q^{j+1} - 2q^j + q^{j-1})}{\Delta t} - \Delta t \nabla_q V(q^j) = 0.$$

# Approximation of kinetic/potential energy in the hydrostatic HPM

$$\mathcal{T} = \frac{1}{2} \sum_{\alpha,\gamma} m_{\alpha,\gamma+1/2} |\dot{x}_{\alpha,\gamma+1/2}|^2.$$

$$\mathcal{V} = \sum_{i,\gamma} \left[ \underbrace{c_v \mu_0 \left( \frac{\mu_{i,\gamma+1/2}}{\mu_0} \right)^{1/(1-\kappa)}}_{\mathcal{V}^p} + \underbrace{g \rho_{i,\gamma+1/2} z_{i,\gamma+1/2}}_{\mathcal{V}^G} \right] \Delta x \Delta z_{i,\gamma+1/2}.$$

# The discrete variational principle leads to

Momentum equation

$$m_{\alpha,\gamma+1/2}\ddot{x}_{\alpha,\gamma+1/2} = F^G_{x,\alpha,\gamma+1/2} + F^P_{x,\alpha,\gamma+1/2},$$

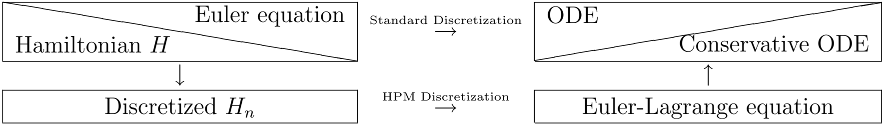
$$F^G_{x,\alpha,\gamma+1/2} = -\frac{\partial \mathcal{V}^G}{\partial x}, \qquad F^P_{x,\alpha,\gamma+1/2} = -\frac{\partial \mathcal{V}^P}{\partial x}$$

Hydrostatic equation

$$0 = F^G_{z,i,\gamma} + F^P_{z,i,\gamma}.$$

Then we find the new vertical positions $z^{n+1}_{i,\gamma+1/2}$ satisfying the discrete hydrostatic balance.

# Advantage of Discrete variational method



# Smoothing Operator

$$H = 1 - \alpha_x^2 \frac{\partial^2}{\partial x^2}$$

over the $x$ grid subject only to $x$ and its periodic boundary conditions.

Here $\alpha_x \geq 0$ is a given smoothing length. Then a smoothed $\widetilde{\mu}_i$ is defined as the solution of

$$\sum_{i'} H_i^{i'} \widetilde{\mu}_{i'} = \overline{\mu}_i.$$

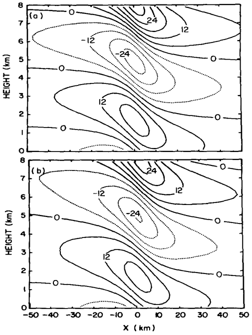
Regularization to smooth out high-frequency waves in a non-diffusive way.

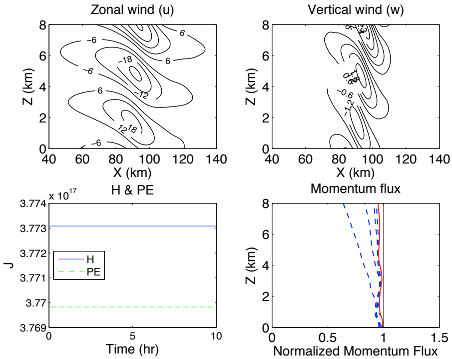
# HPM: A framework for climate simulations

1. Exact mass conservation due to a Lagrangian formulation.
2. Conservative force fields through discrete variational principle.

- Avoiding the ( spurious ) artificial generation of energy
- Exact transportation of conservation properties.
  : The conservation of mass of air, water, and long-lived tracers (Thuburn, 2008).

# Dry simulation of orographic flow in a linear regime




Analytic and Numerical solutions in Durran and Klemp (1983), HPM Simulation

# Moist effect on the stability (Durran and Klemp 1982)

Momentum flux

$$M = \int_{-\infty}^{\infty} \rho u' w' dx,$$

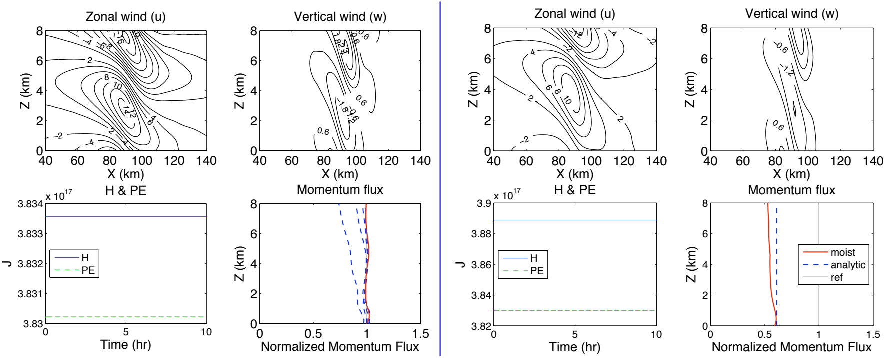
Analytic linear hydrostatic value

$$D = \frac{-\pi}{4} \rho_0 N U h_0^2,$$

Moist stability

$${N_m}^2 = g \left[ \frac{1 + (L q_s / R T)}{1 + (\epsilon L^2 q_s / c_p R T^2)} \left( \frac{d ln\theta}{dz} + \frac{L}{c_p T} \frac{dq_s}{dz} \right) - \frac{dq_w}{dz} \right].$$

# Dry vs Moist simulation of orographic flow in a linear regime



## Summary

- Extension of the HPM method to a hydrostatic vertical slice model.
- Moist processes are newly implemented in the HPM method
- The model captures the main features of mountain waves in a linear regime, and its solutions are in good agreement with analytic solutions
- Numerical models with the HPM method may provide a useful tool for climate studies, where conservative properties are desirable.
- A redistribution of particles will be investigated when diabatic processes lead to a convective instability.