Compact and Stable Discontinuous Galerkin Methods for Convection-Diffusion Problems

S. BRDAR
JOINT WORK WITH R. KLÖFKORN AND A. DEDNER

Int. MetStröm Conference,
June 7, 2011

Berlin

Albert-Ludwigs-Universität Freiburg
UNI FREIBURG

Outline

Motivation

Theoretical results

Numerical results

Summary

DG methods for convection-diffusion problems

Brief history

- Douglas and Dupont 1976: elliptic and parabolic PDEs
- Cockburn, Shu et al 1998: nonlinear parabolic PDE for conservation laws

DG methods for convection-diffusion problems

Brief history

- Douglas and Dupont 1976: elliptic and parabolic PDEs
- Cockburn, Shu et al 1998: nonlinear parabolic PDE for conservation laws

Advantages of DG methods

- easy to achieve higher order without enlarging stencil
- easy construction of discrete function spaces
- easy extension on non-conforming meshes
- efficient parallelization

Brief history

- Douglas and Dupont 1976: elliptic and parabolic PDEs
- Cockburn, Shu et al 1998: nonlinear parabolic PDE for conservation laws

Advantages of DG methods

- easy to achieve higher order without enlarging stencil
- easy construction of discrete function spaces
- easy extension on non-conforming meshes
- efficient parallelization

Disadvantages of DG methods

- high number of unknowns (Navier-Stokes in 3d, 3rd order → 50 unknowns per mesh element)

Stencils of DG methods

- 1^{st} order PDEs: compact stencil even for higher order

Stencils of DG methods

- 1^{st} order PDEs: compact stencil even for higher order
- 2^{nd} order PDEs: compact stencil for higher order?

Stencils of DG methods

- 1^{st} order PDEs: compact stencil even for higher order
- 2^{nd} order PDEs: compact stencil for higher order?
  - Interior Penalty (IP)

    J. Douglas; T. Dupont Interior penalty procedures for elliptic and parabolic Galerkin methods (1976),

    R. Hartmann; P. Houston An optimal order penalty discontinuous Galerkin discretization of the compressible Navier-Stokes equations (2008)

Stencils of DG methods

- 1^{st} order PDEs: compact stencil even for higher order
- 2^{nd} order PDEs: compact stencil for higher order?
  - Interior Penalty (IP)

    J. Douglas; T. Dupont Interior penalty procedures for elliptic and parabolic Galerkin methods (1976),

    R. Hartmann; P. Houston An optimal order penalty discontinuous Galerkin discretization of the compressible Navier-Stokes equations (2008)
  - Bassi-Rebay 2 (BR2)

    F. Bassi; S. Rebay A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations (1997),

    F. Bassi; S. Rebay; G. Mariotti; S. Pedinotti; M. Savini A high-order accurate discontinuous finite element method for invisid turbomachinery flows (1997)

Stencils of DG methods

- 1^{st} order PDEs: compact stencil even for higher order
- 2^{nd} order PDEs: compact stencil for higher order?
  - Interior Penalty (IP)

    J. Douglas; T. Dupont Interior penalty procedures for elliptic and parabolic Galerkin methods (1976),

    R. Hartmann; P. Houston An optimal order penalty discontinuous Galerkin discretization of the compressible Navier-Stokes equations (2008)
  - Bassi-Rebay 2 (BR2)

    F. Bassi; S. Rebay A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations (1997),

    F. Bassi; S. Rebay; G. Mariotti; S. Pedinotti; M. Savini A high-order accurate discontinuous finite element method for invisid turbomachinery flows (1997)
  - Local Discontinuous Galerkin (LDG) !!!

    B. Cockburn; C.-W. Shu The Local Discontinuous Galerkin method for time-dependent convection-diffusion problems (1998)

Stencils of DG methods

- 1^{st} order PDEs: compact stencil even for higher order
- 2^{nd} order PDEs: compact stencil for higher order?
  - Interior Penalty (IP)

    J. Douglas; T. Dupont Interior penalty procedures for elliptic and parabolic Galerkin methods (1976),

    R. Hartmann; P. Houston An optimal order penalty discontinuous Galerkin discretization of the compressible Navier-Stokes equations (2008)
  - Bassi-Rebay 2 (BR2)

    F. Bassi; S. Rebay A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations (1997),

    F. Bassi; S. Rebay; G. Mariotti; S. Pedinotti; M. Savini A high-order accurate discontinuous finite element method for invisid turbomachinery flows (1997)
  - Local Discontinuous Galerkin (LDG) !!!

    B. Cockburn; C.-W. Shu The Local Discontinuous Galerkin method for time-dependent convection-diffusion problems (1998)
  - Compact Discontinuous Galerkin (CDG)

    J. Peraire; P.-O. Persson The Compact Discontinuous Galerkin (CDG) method for elliptic problems (2008)

Elliptic case.

The Poisson equation

$$
\begin{aligned}
-\Delta u &= s && \text{in } \Omega, \\
u &= g_D && \text{on } \partial\Omega,
\end{aligned}
$$

$\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ a bounded polygon, $s \in L^d(\Omega)$, $g_D \in L^d(\partial\Omega)$, and $u : \mathbb{R}^d \to \mathbb{R}$.

Elliptic case.

The Poisson equation

$$\boldsymbol{\sigma} = \nabla u, \quad -\nabla \cdot \boldsymbol{\sigma} = s \quad \text{in} \quad \Omega,$$
$$u = g_D \quad \text{on} \quad \partial\Omega,$$

$\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ a bounded polygon, $s \in L^d(\Omega)$, $g_D \in L^d(\partial\Omega)$, and $u : \mathbb{R}^d \to \mathbb{R}$.

UNI FREIBURG

Elliptic case.

The Poisson equation

$$\begin{aligned} \boldsymbol{\sigma} = \nabla u, \quad -\nabla \cdot \boldsymbol{\sigma} &= s \quad \text{in} \quad \Omega, \\ u &= g_D \quad \text{on} \quad \partial\Omega, \end{aligned}$$

$\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ a bounded polygon, $s \in L^d(\Omega)$, $g_D \in L^d(\partial\Omega)$, and $u : \mathbb{R}^d \to \mathbb{R}$.

Let φ and ψ be test functions and $K \subset \Omega$

$$\begin{aligned} (\boldsymbol{\sigma}, \psi)_K &= -(u, \nabla \cdot \psi)_K + \langle u, \boldsymbol{n}_K \cdot \psi \rangle_{\partial K}, \\ (\boldsymbol{\sigma}, \nabla\varphi)_K &= (s, \varphi)_K + \langle \boldsymbol{\sigma} \cdot \boldsymbol{n}_K, \varphi \rangle_{\partial K} \end{aligned}$$

DG flux formulation

$$\begin{aligned}(\boldsymbol{\sigma}, \psi)_K &= -(u, \nabla \cdot \psi)_K + \langle u, \boldsymbol{n}_K \cdot \psi \rangle_{\partial K}, \\ (\boldsymbol{\sigma}, \nabla \varphi)_K &= (s, \varphi)_K + \langle \boldsymbol{\sigma} \cdot \boldsymbol{n}_K, \varphi \rangle_{\partial K}\end{aligned}$$

For a given partition $\mathcal{T}_h$ into polygons we define

$$\mathcal{P}_k^m(\mathcal{T}_h) = \{\boldsymbol{v} \ : \ \boldsymbol{v}|_K \in [\mathcal{P}_k(K)]^m, \ K \in \mathcal{T}_h\} \quad \text{for } m \in \mathbb{N}.$$

DG flux formulation

$$
\begin{aligned}
(\boldsymbol{\sigma}, \psi)_K &= -(u, \nabla \cdot \psi)_K + \langle u, \boldsymbol{n}_K \cdot \psi \rangle_{\partial K}, \\
(\boldsymbol{\sigma}, \nabla \varphi)_K &= (s, \varphi)_K + \langle \boldsymbol{\sigma} \cdot \boldsymbol{n}_K, \varphi \rangle_{\partial K}
\end{aligned}
$$

For a given partition $\mathcal{T}_h$ into polygons we define

$$
\mathcal{P}_k^m(\mathcal{T}_h) = \{ \boldsymbol{v} \,:\, \boldsymbol{v}|_K \in [\mathcal{P}_k(K)]^m,\ K \in \mathcal{T}_h \} \quad \text{for } m \in \mathbb{N}.
$$

DG flux formulation

Find $u_h \in \mathcal{P}_k(\mathcal{T}_h)$ and $\boldsymbol{\sigma}_h \in \mathcal{P}_k^d(\mathcal{T}_h)$ so that

$$
\begin{aligned}
(\boldsymbol{\sigma}_h, \psi)_K &= -(u_h, \nabla \cdot \psi)_K + \langle u_h, \boldsymbol{n}_K \cdot \psi \rangle_{\partial K}, \\
(\boldsymbol{\sigma}_h, \nabla \varphi)_K &= (s, \varphi)_K + \langle \boldsymbol{\sigma}_h \cdot \boldsymbol{n}_K, \varphi \rangle_{\partial K}
\end{aligned}
$$

for all $\varphi \in \mathcal{P}_k(K)$, $\psi \in \mathcal{P}_k^d(K)$, and $K \in \mathcal{T}_h$.

DG flux formulation

$$\begin{aligned} (\boldsymbol{\sigma}, \psi)_K &= -(u, \nabla \cdot \psi)_K + \langle u, \boldsymbol{n}_K \cdot \psi \rangle_{\partial K}, \\ (\boldsymbol{\sigma}, \nabla \varphi)_K &= (s, \varphi)_K + \langle \boldsymbol{\sigma} \cdot \boldsymbol{n}_K, \varphi \rangle_{\partial K} \end{aligned}$$

For a given partition $\mathcal{T}_h$ into polygons we define

$$\mathcal{P}_k^m(\mathcal{T}_h) = \{\boldsymbol{v} \ : \ \boldsymbol{v}|_K \in [\mathcal{P}_k(K)]^m, \ K \in \mathcal{T}_h\} \quad \text{for } m \in \mathbb{N}.$$

DG flux formulation

Find $u_h \in \mathcal{P}_k(\mathcal{T}_h)$ and $\boldsymbol{\sigma}_h \in \mathcal{P}_k^d(\mathcal{T}_h)$ so that

$$\begin{aligned} (\boldsymbol{\sigma}_h, \psi)_K &= -(\nabla u_h, \psi)_K + \langle u_h - \hat{u}, \boldsymbol{n}_K \cdot \psi \rangle_{\partial K}, \\ (\boldsymbol{\sigma}_h, \nabla \varphi)_K &= (s, \varphi)_K + \langle \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n}_K, \varphi \rangle_{\partial K} \end{aligned}$$

for all $\varphi \in \mathcal{P}_k(K)$, $\psi \in \mathcal{P}_k^d(K)$, and $K \in \mathcal{T}_h$.

DG flux formulation

$$\begin{aligned}(\boldsymbol{\sigma},\psi)_K &= -(u,\nabla\cdot\psi)_K + \langle u, \boldsymbol{n}_K\cdot\psi\rangle_{\partial K},\\ (\boldsymbol{\sigma},\nabla\varphi)_K &= (s,\varphi)_K + \langle \boldsymbol{\sigma}\cdot\boldsymbol{n}_K,\varphi\rangle_{\partial K}\end{aligned}$$

For a given partition $\mathcal{T}_h$ into polygons we define

$$\mathcal{P}_k^m(\mathcal{T}_h) = \{\boldsymbol{v} \,:\, \boldsymbol{v}|_K \in [\mathcal{P}_k(K)]^m,\ K\in\mathcal{T}_h\} \quad \text{for } m\in\mathbb{N}.$$

DG flux formulation

Find $u_h \in \mathcal{P}_k(\mathcal{T}_h)$ and $\boldsymbol{\sigma}_h \in \mathcal{P}_k^d(\mathcal{T}_h)$ so that

$$\begin{aligned}(\boldsymbol{\sigma}_h,\psi)_K &= -(\nabla u_h,\psi)_K + \langle u_h-\hat{u}, \boldsymbol{n}_K\cdot\psi\rangle_{\partial K},\\ (\boldsymbol{\sigma}_h,\nabla\varphi)_K &= (s,\varphi)_K + \langle \hat{\boldsymbol{\sigma}}\cdot\boldsymbol{n}_K,\varphi\rangle_{\partial K}\end{aligned}$$

for all $\varphi\in\mathcal{P}_k(K)$, $\psi\in\mathcal{P}_k^d(K)$, and $K\in\mathcal{T}_h$.

Can we avoid computation of the intermediate variable $\boldsymbol{\sigma}$?

DG primal formulation 1

DG flux formulation

Find $u_h \in \mathcal{P}_k(\mathcal{T}_h)$ and $\boldsymbol{\sigma}_h \in \mathcal{P}_k^d(\mathcal{T}_h)$ so that

$$\begin{aligned}(\boldsymbol{\sigma}_h, \psi)_K &= -(u_h, \nabla \cdot \psi)_K + \langle \hat{u}, \boldsymbol{n}_K \cdot \psi \rangle_{\partial K}, \\ (\boldsymbol{\sigma}_h, \nabla \varphi)_K &= (s, \varphi)_K + \langle \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n}_K, \varphi \rangle_{\partial K}\end{aligned}$$

for all $\varphi \in \mathcal{P}_k(K)$, $\psi \in \mathcal{P}_k^d(K)$, and $K \in \mathcal{T}_h$.

Can we avoid computation of the intermediate variable $\boldsymbol{\sigma}$?
Yes, if $\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}(u_h)$ (i.e. Arnold et al 2002).

DG primal formulation 1

DG flux formulation

Find $u_h \in \mathcal{P}_k(\mathcal{T}_h)$ and $\boldsymbol{\sigma}_h \in \mathcal{P}_k^d(\mathcal{T}_h)$ so that

$$
\begin{aligned}
(\boldsymbol{\sigma}_h, \psi)_K &= -(u_h, \nabla \cdot \psi)_K + \langle \hat{u}, \boldsymbol{n}_K \cdot \psi \rangle_{\partial K}, \\
(\boldsymbol{\sigma}_h, \nabla \varphi)_K &= (s, \varphi)_K + \langle \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n}_K, \varphi \rangle_{\partial K}
\end{aligned}
$$

for all $\varphi \in \mathcal{P}_k(K)$, $\psi \in \mathcal{P}_k^d(K)$, and $K \in \mathcal{T}_h$.

Can we avoid computation of the intermediate variable $\boldsymbol{\sigma}$?
Yes, if $\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}(u_h)$ (i.e. Arnold et al 2002).

DG primal formulation

Find $u_h \in \mathcal{P}_k(\mathcal{T}_h)$ so that $B(u_h, \varphi) = (s, \varphi)_\Omega$ for all $\varphi \in \mathcal{P}_k(\mathcal{T}_h)$ with

$$
\begin{aligned}
B(u_h, \varphi) &= (\nabla u_h, \nabla \psi)_\Omega - \sum_{e \in \Gamma} \langle [\![u_h]\!], \{\!\{ \nabla \varphi \}\!\} \rangle_e + \langle \{\!\{ \hat{\boldsymbol{\sigma}} \}\!\}, [\![\varphi]\!] \rangle_e \\
&\quad + \sum_{e \in \Gamma_i} \langle \{ \hat{u} - u_h \}, [\nabla \varphi] \rangle_e - \langle [\hat{\boldsymbol{\sigma}}], \{ \varphi \} \rangle_e \, .
\end{aligned}
$$

DG primal formulation 2

DG primal formulation

Find $u_h \in \mathcal{P}_k(\mathcal{T}_h)$ so that $B(u_h, \varphi) = (s, \varphi)_\Omega$ for all $\varphi \in \mathcal{P}_k(\mathcal{T}_h)$ with

$$B(u_h, \varphi) = (\nabla u_h, \nabla \psi)_\Omega - \sum_{e \in \Gamma} \langle [\![u_h]\!], \{\!\{ \nabla \varphi \}\!\} \rangle_e + \langle \{\!\{ \hat{\boldsymbol{\sigma}} \}\!\}, [\![\varphi]\!] \rangle_e$$

$$+ \sum_{e \in \Gamma_i} \langle \{ \hat{u} - u_h \}, [\nabla \varphi] \rangle_e - \langle [\hat{\boldsymbol{\sigma}}], \{ \varphi \} \rangle_e \,.$$

DG primal formulation 2

DG primal formulation

Find $u_h \in \mathcal{P}_k(\mathcal{T}_h)$ so that $B(u_h, \varphi) = (s, \varphi)_\Omega$ for all $\varphi \in \mathcal{P}_k(\mathcal{T}_h)$ with

$$
\begin{aligned}
B(u_h, \varphi) \quad = \quad & (\nabla u_h, \nabla \psi)_\Omega - \sum_{e \in \Gamma} \langle [\![u_h]\!], \{\!\{ \nabla \varphi \}\!\} \rangle_e + \langle \{\!\{ \hat{\boldsymbol{\sigma}} \}\!\}, [\![\varphi]\!] \rangle_e \\
& + \sum_{e \in \Gamma_i} \langle \{ \hat{u} - u_h \}, [\nabla \varphi] \rangle_e - \langle [\hat{\boldsymbol{\sigma}}], \{ \varphi \} \rangle_e \, .
\end{aligned}
$$

Jump and average. Let $K_e^-, K_e^+ \in \mathcal{T}_h$ and $e \in \Gamma_i$

$$
\begin{aligned}
[\![\phi]\!]_e &= (\varphi|_{K_e^-} - \varphi|_{K_e^+}) \boldsymbol{n}_{K_e^-}, & [\boldsymbol{\xi}]_e &= (\boldsymbol{\xi}|_{K_e^-} - \boldsymbol{\xi}|_{K_e^+}) \cdot \boldsymbol{n}_{K_e^-} \\
\{ \phi \}_e &= 1/2 \, (\varphi|_{K_e^-} + \varphi|_{K_e^+}), & \{\!\{ \boldsymbol{\xi} \}\!\}_e &= 1/2 \, (\boldsymbol{\xi}|_{K_e^-} + \boldsymbol{\xi}|_{K_e^+}) \, .
\end{aligned}
$$

If $e \in \Gamma \setminus \Gamma_i$

$$
\begin{aligned}
[\![\phi]\!]_e &= (\varphi|_{K_e} - g_D') \boldsymbol{n}_{K_e}, & [\boldsymbol{\xi}]_e &= \boldsymbol{\xi}|_{K_e} \cdot \boldsymbol{n}_{K_e} \\
\{ \phi \}_e &= \varphi|_{K_e}, & \{\!\{ \boldsymbol{\xi} \}\!\}_e &= \boldsymbol{\xi}|_{K_e} \, .
\end{aligned}
$$

DG methods

Method	$\hat{u}$	$\hat{\boldsymbol{\sigma}}$
IP	$\{u_h\}$	$\{\!\{\nabla u_h\}\!\} - \eta[\![u_h]\!]$
BR2	$\{u_h\}$	$\{\!\{\nabla u_h\}\!\}$ $+\chi\{\!\{\boldsymbol{r}_e([\![u_h]\!])\}\!\}$
CDG	$\{u_h\} - \boldsymbol{\beta}_e \cdot [\![u_h]\!]$	$\{\!\{\nabla u_h\}\!\} - \eta[\![u_h]\!] + \boldsymbol{\beta}_e[\nabla u_h]$ $+\chi\big(\{\!\{\boldsymbol{L}_e(u_h)\}\!\} + \boldsymbol{\beta}_e[\boldsymbol{L}_e(u_h)]\big)$
CDG2	$\{u_h\}$	$\{\!\{\nabla u_h\}\!\} - \eta[\![u_h]\!]$ $+\chi\big(\{\!\{\boldsymbol{L}_e(u_h)\}\!\} + \boldsymbol{\beta}_e[\boldsymbol{L}_e(u_h)]\big)$

Parameter-free. Determine the penalty factor η and the lifting factor χ.

DG methods

Method	$\hat{u}$	$\hat{\boldsymbol{\sigma}}$
IP	$\{u_h\}$	$\{\!\{\nabla u_h\}\!\} - \eta[\![u_h]\!]$
BR2	$\{u_h\}$	$\{\!\{\nabla u_h\}\!\}$ $+\chi\{\!\{\boldsymbol{r}_e([\![u_h]\!])\}\!\}$
CDG	$\{u_h\} - \boldsymbol{\beta}_e \cdot [\![u_h]\!]$	$\{\!\{\nabla u_h\}\!\} - \eta[\![u_h]\!] + \boldsymbol{\beta}_e[\nabla u_h]$ $+\chi\big(\{\!\{\boldsymbol{L}_e(u_h)\}\!\} + \boldsymbol{\beta}_e[\boldsymbol{L}_e(u_h)]\big)$
CDG2	$\{u_h\}$	$\{\!\{\nabla u_h\}\!\} - \eta[\![u_h]\!]$ $+\chi\big(\{\!\{\boldsymbol{L}_e(u_h)\}\!\} + \boldsymbol{\beta}_e[\boldsymbol{L}_e(u_h)]\big)$

Lifting. $\boldsymbol{r}_e,\ \boldsymbol{L}_e : [L^2(e)]^d \to \mathcal{P}_k^m(\mathcal{T}_h)$ and $\boldsymbol{l}_e : L^2(e) \to \mathcal{P}_k^m(\mathcal{T}_h)$ given as

$$(\boldsymbol{r}_e(\boldsymbol{\xi}), \boldsymbol{\tau})_\Omega = -\langle \boldsymbol{\xi}, \{\!\{\boldsymbol{\tau}\}\!\}\rangle_e, \quad (\boldsymbol{l}_e(\phi), \boldsymbol{\tau})_\Omega = -\langle \psi, [\boldsymbol{\tau}]\rangle_e,$$

$$\boldsymbol{L}_e(\boldsymbol{\xi}) = \boldsymbol{r}_e(\boldsymbol{\xi}) + \boldsymbol{l}_e(\boldsymbol{\beta}_e \cdot \boldsymbol{\xi})$$

for all $\boldsymbol{\xi}, \boldsymbol{\tau} \in \mathcal{P}_k^m(\mathcal{T}_h)$ and $\phi \in \mathcal{P}_k(\mathcal{T}_h)$.

Switch. $\boldsymbol{\beta}_e$ induces ordering between adjacent grid elements.

$$\boldsymbol{\beta}_e = \boldsymbol{n}_{K_e^-}/2 = -\boldsymbol{n}_{K_e^+}/2$$

For CDG choose any $\boldsymbol{w} \in \mathbb{R}^d$ such that $\boldsymbol{w} \cdot \boldsymbol{n}_e \neq 0,\ \forall\, e$ the upwind switch is determined by $\boldsymbol{w} \cdot \boldsymbol{n}_{K_e^-} > 0$.

A-priori estimate:

For stable, bounded, consistent, and adjoint consistent methods the discrete solution $u_h \in V_h$ of a linear elliptic problem $(-\Delta u = 0)$ is estimated by

$$\|u - u_h\|_\Omega \leq Ch^{k+1} |u|_{k+1,\Omega},$$

- C is a constant
- k is the polynomial degree of the basis functions from V_h.
- All the methods studied here fall into this category

Coercivity for linear elliptic problem

Theorem (Coercivity)

Let each $K \in \mathcal{T}_h$ be an image of a fixed reference element $\hat{K}$ under an affine mapping.
The BR2, CDG, CDG2 method are coercive if one of the following conditions is fulfilled:

a) η *is chosen sufficiently large and* $\chi \geq 0$.

b) $\eta \geq 0$ *and* $\chi > \chi_0$, *where*

- $\chi_0 = N_{\mathcal{T}_h}$ *for BR2,*
- $\chi_0 = N_{\mathcal{T}_h}^{out}$ *for CDG,*
- *and for CDG2*

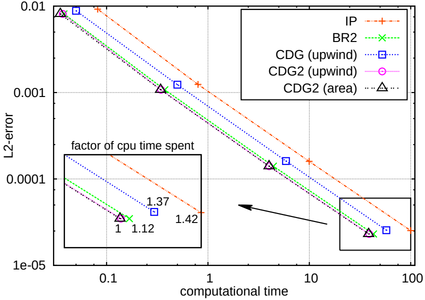
$$\chi_0 = \frac{N_{\mathcal{T}_h}}{4}\big(1 + \nu(\boldsymbol{\beta})\big)$$

with $\nu(\boldsymbol{\beta}) = \max_{e \in \Gamma_i}\{|K_e^-|/|K_e^+|\}$ *and* K_e^-, K_e^+ *determined by* $\boldsymbol{\beta}$.

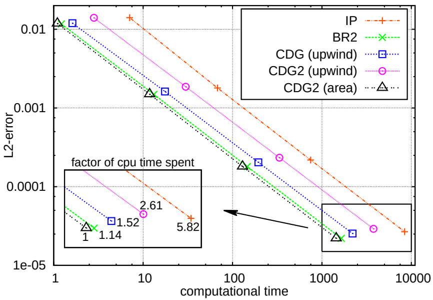
$N_{\mathcal{T}_h}$ *is the maximal number of interfaces one element can have.*
$N_{\mathcal{T}_h}^{out}$ *is the maximal number of outflow interfaces one element can have w.r.t. the upwind vector.*

Area switch

For CDG2 choose area switch to fullfill $|\boldsymbol{n}_{K_e^-}| \leq |\boldsymbol{n}_{K_e^+}|$.



structured grid



unstructured grid

BR2 vs CDG2 in linear case

Lemma (BR2 and CDG2 on special grids)

Under the conditions of the Coercivity Theorem BR2 and CDG2 coincide on grids $\mathcal{T}_h$ where each element is isometric to another if

$$\chi_{BR2} = 2 \cdot \chi_{CDG2} \, .$$

Remark (BR2 and CDG2 on general grids)

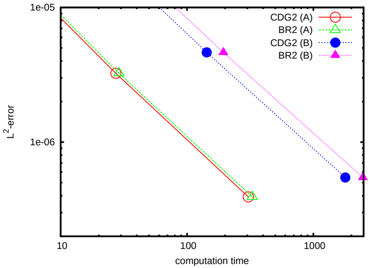
Consider the conditions of the Coercivity Theorem.

$$\chi_{BR2} = 2 \cdot \chi_{CDG2} + \frac{N_{\mathcal{T}_h}}{2} (1 - \nu(\boldsymbol{\beta}_e)) \, .$$

Remark (Evaluation of lifting operators)

BR2 requires evaluation of the lifting operator $\boldsymbol{r}_e$ on both elements containing the edge e, whereas CDG2 requires evaluation of $\boldsymbol{L}_e$ on only one element.

BR2 vs CDG2 for non-affine grid mappings



A

Scheme	CPU time	L^2-error
CDG2	305	3.91e-07
BR2	329	3.92e-07

B

Scheme	CPU time	L^2-error
CDG2	1790	5.47e-07
BR2	2469	5.46e-07

Figure: Comparison on affine (A) and non-affine (B) quadrilateral grids. Problem is on the quadrilateral domain with corners $(0,0),(1,0),(1,1),(0,1)$ (A) and $(0.4,0),(1,0),(1,1.4),(0.1)$ (B). The graph (left) contains level 4 and 5 of the simulation cycle and the table (right) contains the numbers of the level 5 run.

CDG coercivity for $\chi = 1$?

In Peraire and Persson (2008) CDG has $\chi = 1$.

Counterexample in 3D

In

Eymard, Henry, Herbin, Hubert, Klöfkorn, and Manzini
3D Benchmark on Discretization Schemes for Anisotropic Diffusion Problems on General Grids (2011)

we get for Test Case 1 using grid **`tet.0.msh`** that the minimal eigenvalue of the stiffness matrix of the bilinear form $B(u_h, u_h)$ is -12.167.

With theoretical values minimal eigenvalue is positive.

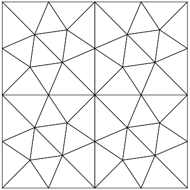
Elliptic problem

Herbin, Hubert
Benchmark on discretization schemes for anisotropic diffusion problems on general grids, FVCA, London (2008)

$$-\nabla \cdot (A\nabla u) = f \quad \text{on } \Omega = [0,1]^2,$$
$$u = g \quad \text{on } \partial\Omega,$$
$$A = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad \varepsilon = 10^3,$$

with analytical solution $u = g$, where

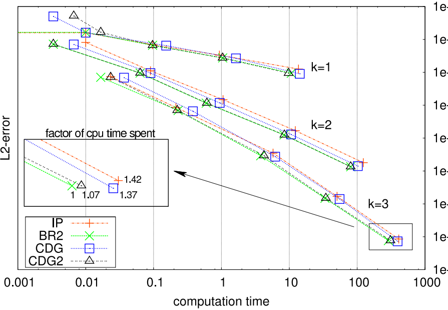
$$g(x,y) = \sin(2\pi x) e^{-2\pi\sqrt{1/\varepsilon y}}.$$



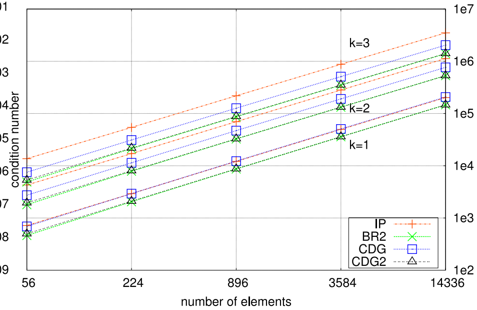
- **mesh1** of the benchmark,
- CG solver is used,
- IP uses η of Ainsworth, Rankin (2009).

Efficiency

$k = 1, 2, 3,\ \chi_{\text{BR2}} = 3,\ \chi_{\text{CDG}} = 2,\ \chi_{\text{CDG2}} = 1.5$



efficiency



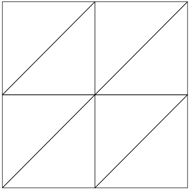
condition number

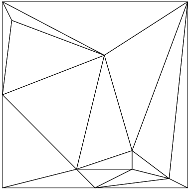
Linear advection-diffusion.

$$
\begin{aligned}
\partial_t u + \nabla \cdot (u\boldsymbol{v}) - \varepsilon \Delta u &= 0 \qquad \text{on } \Omega = (0,1)^2 \times (0, 0.1),\\
u &= g_D \qquad \text{on } \partial\Omega,
\end{aligned}
$$

where $\varepsilon = 0.1$ and

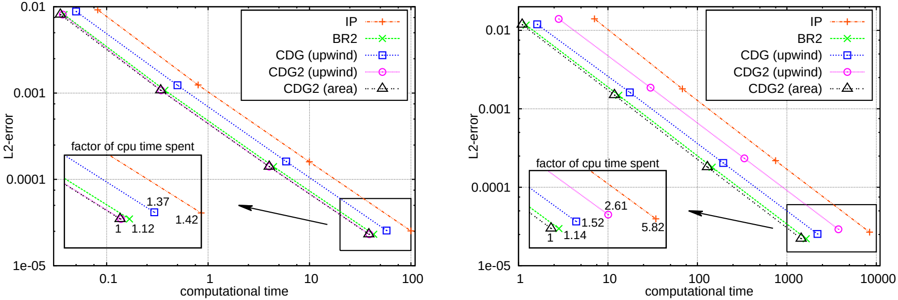
$$
\begin{aligned}
\boldsymbol{v} &= (0.1, 0.2)\\
g_D &= g_{D,1} + g_{D,2}\\
g_{D,1} &= 0.6\cos(2\pi x) + 0.8\sin(2\pi x) + 1.2\cos(\pi x) + 0.4\sin(\pi x)\\
g_{D,2} &= 0.9\cos(0.7\pi x) + 0.2\sin(0.7\pi x) + 0.3\cos(0.5\pi x) + 0.1\sin(0.5\pi x)\\
u(x,y,t) &= e^{-3\pi\varepsilon t} g_{D,1}(x,y) + e^{-1.2\pi\varepsilon t} g_{D,2}(x,y)
\end{aligned}
$$





Linear advection-diffusion.

$k = 2$ with 3rd order semi-implicit YZ(3,3) of Yoh and Zhong

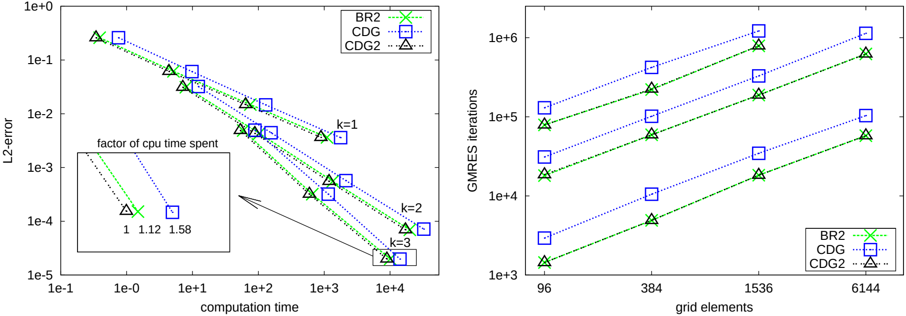


structured grid

unstructured grid

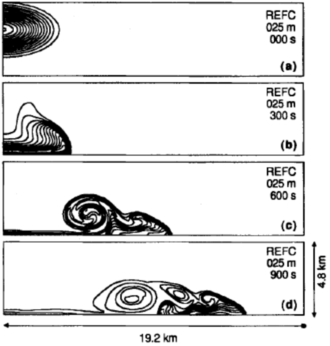
Nonlinear convection-diffusion systems

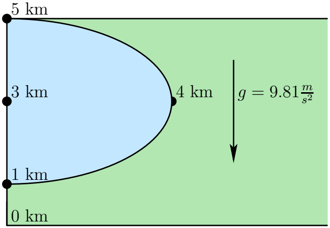
Navier-Stokes equations,
Analytical solution: C^∞ solution from Lorcher, Gassner, Munz (2008),
Grid: Unstructured triangular



CDG2 for an atmospheric test case

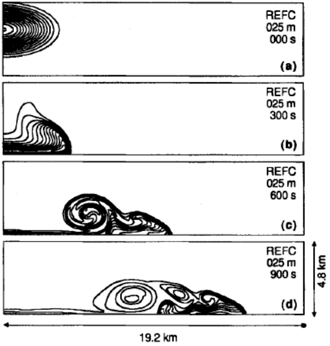
Navier-Stokes equations in θ-form (pot. temperature),
Reference solution: Straka et al. (1993),

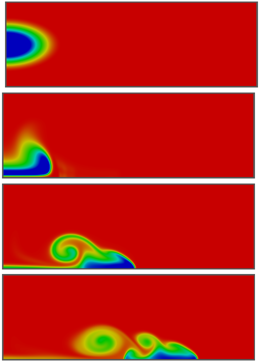




CDG2 for an atmospheric test case

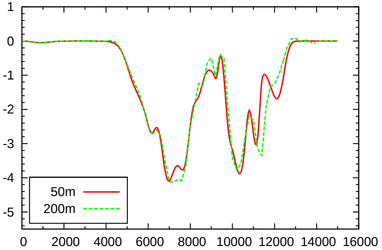
Navier-Stokes equations in θ-form (pot. temperature),
Reference solution: Straka et al. (1993),

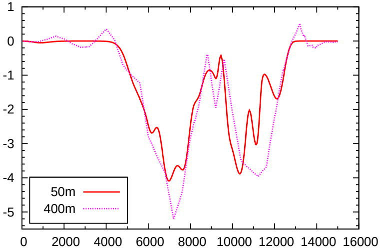




Convergence study: Straka's test case

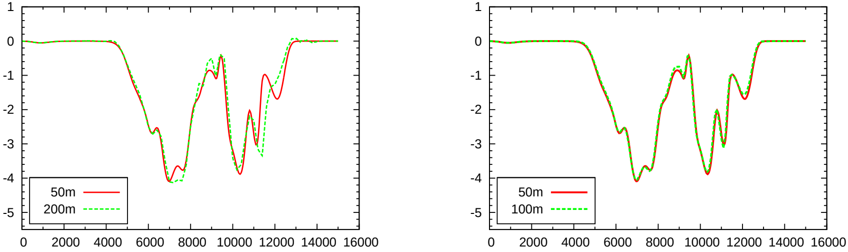
Navier-Stokes equations in θ-form (pot. temperature),
Reference solution: Straka et al. (1993),





Convergence study: Straka's test case

Navier-Stokes equations in θ-form (pot. temperature),
Reference solution: Straka et al. (1993),



Summary

- overview on DG method for convection-diffusion problems

Summary

- overview on DG method for convection-diffusion problems
- new parameter free method (CDG2) presented

Summary

- overview on DG method for convection-diffusion problems
- new parameter free method (CDG2) presented
- stability proven for linear heat equation

Summary

- overview on DG method for convection-diffusion problems
- new parameter free method (CDG2) presented
- stability proven for linear heat equation
- improved stability of CDG in the linear case

Summary

- overview on DG method for convection-diffusion problems
- new parameter free method (CDG2) presented
- stability proven for linear heat equation
- improved stability of CDG in the linear case
- numerical results show the performance of the method

Summary

- overview on DG method for convection-diffusion problems
- new parameter free method (CDG2) presented
- stability proven for linear heat equation
- improved stability of CDG in the linear case
- numerical results show the performance of the method

B., Dedner, Klöfkorn.
Compact and stable Discontinuous Galerkin methods for convection-diffusion problems, Preprint 2010, Math. Institute, Uni Freiburg

Summary

- overview on DG method for convection-diffusion problems
- new parameter free method (CDG2) presented
- stability proven for linear heat equation
- improved stability of CDG in the linear case
- numerical results show the performance of the method

B., Dedner, Klöfkorn.
Compact and stable Discontinuous Galerkin methods for convection-diffusion problems, Preprint 2010, Math. Institute, Uni Freiburg

Thank you for your attention!

UNI FREIBURG