Subgrid scale parameterization based on stochastic mode reduction

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Outline

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1. Motivation
2. Stochastic mode reduction for the forced Burgers equation
3. Inviscid Burgers equation

Motivation

- large range of temporal and spatial scales in the atmosphere
- some parts of the spectrum are of interest, e.g. low-frequency, large-scale modes
- need for conceptual models capturing the essential processes
- numerically efficient reduced models: ensemble simulations, long-term simulations of the coupled atmosphere-ocean system

Strategies for the construction of reduced models

- Multiple scales asymptotic approach (Klein, 2000; Majda & Klein, 2003)
- Projection on some suitable basis functions (Kasahara, 1977; Selten 1995; Kwasniok, 1996; Achatz & Branstator, 1999; Delsole, 2000)
- Empirical fitting approach (Whitaker & Sardeshmukh, 1998; Branstator & Haupt, 1998)
- ...

MTV stochastic mode reduction strategy

A. Majda, I. Timofeyev, E. Vanden-Eijnden
A mathematical framework for stochastic climate models
Commun. Pure Appl. Math., 2001

A. Majda, I. Timofeyev, E. Vanden-Eijnden
A priori tests of a stochastic mode reduction strategy
Physica D, 2002

A. Majda, I. Timofeyev, E. Vanden-Eijnden
Systematic strategies for stochastic mode reduction in climate
J. Atmos, Sci, 2003

C. Franzke, A. Majda, E. Vanden-Eijnden
Low-order stochastic mode reduction for a realistic barotropic model climate
J. Atmos. Sci., 2005

C. Franzke and A. Majda
Low-order stochastic mode reduction for a prototype atmospheric GCM
J. Atmos. Sci., 2006

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Forced Burgers equation

We consider the Burgers equation with forcing and dissipation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2} - \nu \frac{\partial u}{\partial x}\right) = f(x,t)\,,$$

where $f(x,t) = \sum_{k=1}^{k_c} \alpha \frac{A}{\sqrt{k\Delta t}} \cos\left(kx + \phi\right)$, with α, ϕ random numbers.

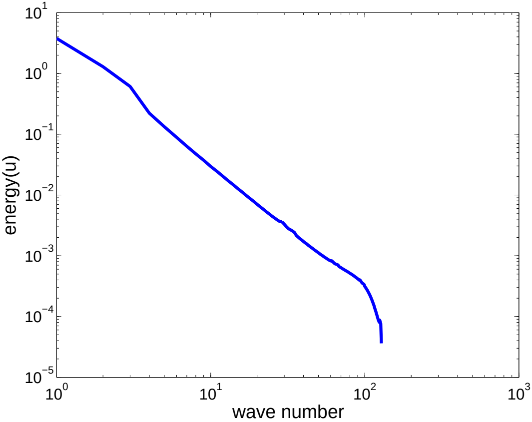
Using an energy and momentum conserving discretization scheme (Kruskal-Zabusky), the discrete form reads

$$\frac{d}{dt}u_i + \frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta_x} = f_i\,,$$

where

$$F_{i+\frac{1}{2}} = \frac{1}{6}\left(u_{i+1}^2 + u_i u_{i+1} + u_i^2\right) - \nu \frac{u_{i+1} - u_i}{\Delta_x}\,.$$

Forced Burgers equation



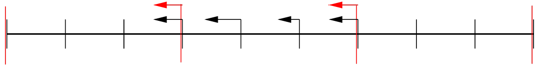
Spectrum of u from a simulation with $k_c = 3\,, \nu = 2 \times 10^{-2}$ and a resolution of 256 points.

Resolved and unresolved modes

The domain is split into intervals of size $n\Delta_x$. We define a mean value x and a deviation y

$$x_{\hat{i}} = \frac{1}{n} \sum_{k=n\hat{i}}^{n(\hat{i}+1)-1} u_k \,,$$

$$y_i = u_i - x_{\hat{i}} \,.$$

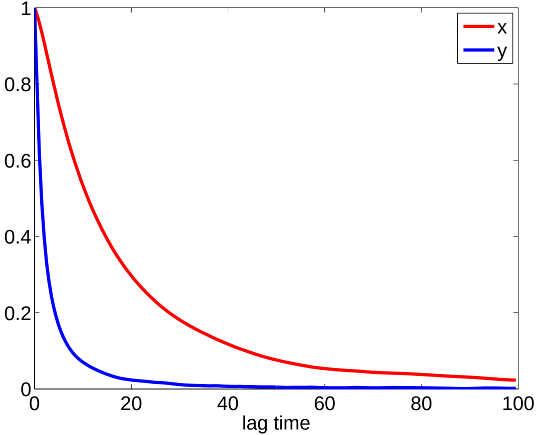


We can write for the Burgers equation

$$\frac{d}{dt} x_{\hat{i}} + \frac{F_{n(\hat{i}+1)-\frac{1}{2}} - F_{n\hat{i}-\frac{1}{2}}}{n\Delta_x} = F^x_{\hat{i}} \,,$$

$$\frac{d}{dt} y_i + \frac{F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}}{\Delta_x} - \frac{F_{n(\hat{i}+1)-\frac{1}{2}} - F_{n\hat{i}-\frac{1}{2}}}{n\Delta_x} = F^y_i \,.$$

Fast and slow modes



Time autocorrelation function of x and y from a simulation with 256 grid points and $n = 8$.

Stochastic mode reduction

The discretized Burgers equation can be written in the following abstract form

$$\dot{x}_i = B^{xxx}_{ijk} x_j x_k + B^{xxy}_{ijk} x_j y_k + B^{xyy}_{ijk} y_j y_k + L^{xy}_{ij} y_j + L^{xx}_{ij} x_j + F^x_i \,,$$
$$\dot{y}_i = B^{yxx}_{ijk} x_j x_k + B^{yxy}_{ijk} x_j y_k + B^{yyy}_{ijk} y_j y_k + L^{yy}_{ij} y_j + L^{yx}_{ij} x_j + F^y_i \,.$$

The stochastic mode reduction procedure assumes that the term $B^{yyy}_{ijk} y_j y_k + L^{yy}_{ij} y_j$ can be represented as an Ornstein-Uhlenbeck process.

Find the OU process

The values of y_i inside each coarse cell are transformed into Fourier space

$$\hat{y}_i = T_{ij} y_j \, ,$$
$$y_j = T^{-1}_{ji} \hat{y}_i \, .$$

The term $B^{yyy} yy + L^{yy} y$ is modeled in Fourier space as an OU process in $\hat{y}$

$$T_{ij} \left(B^{yyy}_{jkl} y_k y_l + L^{yy}_{jk} y_k \right) \approx \Gamma_{ij} \hat{y}_j + \sigma_i \dot{W}_i$$

Γ and σ can be determined using the maximum-likelihood-approach

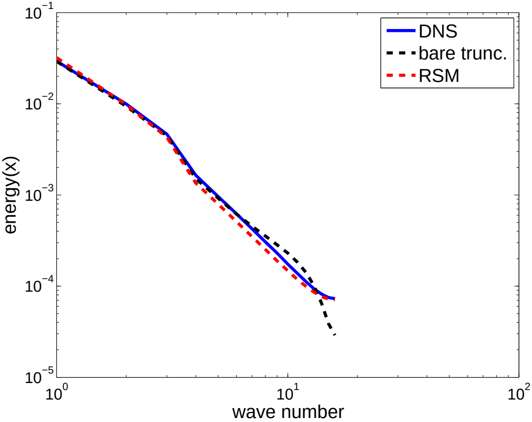
- same OU parameters for each coarse grid cell
- Γ has complex eigenvalues with negative real parts.
- generalization of the MTV stochastic mode reduction strategy for an oscillating OU process

The reduced stochastic model

Applying a generalized stochastic mode reduction procedure, the subgrid scale modes are eliminated. We obtain the following effective stochastic differential equation for x

$$
\begin{aligned}
dx_i(t) = & \sum_{j,k=i-1}^{i+1} B_{ijk}^{xxx} x_j x_k dt + \sum_{j=i-1}^{i+1} L_{ij}^{xx} x_j dt + F_i^x dt \\
& + \sum_{j=i-1}^{i+1} M_{ij} x_j dt + \sum_{j,k,l=i-2}^{i+2} C_{ijkl} x_j x_k x_l dt \\
& + \sum_{j=i-1}^{i+1} P_{ij} dW_j + \sum_{k=i-1}^{i+1} \left(N_{ik} + \sum_{j=i-1}^{i+1} D_{ijk} x_j \right) dW_k \, .
\end{aligned}
$$

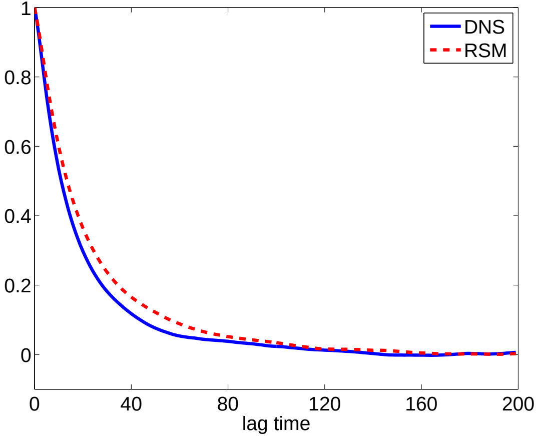
The reduced stochastic model



Spectrum of x from DNS (256 points), reduced stochastic model (32 points) and bare truncation (32 points).

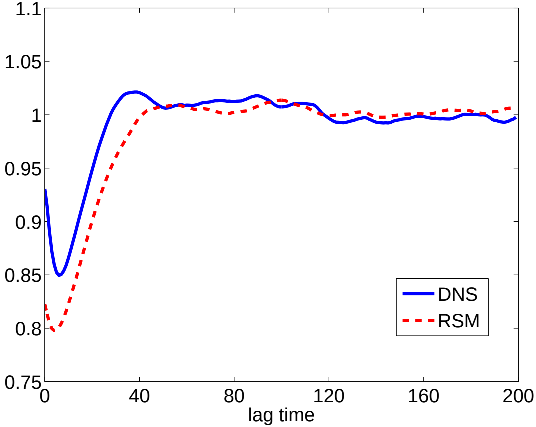
The reduced stochastic model

Time autocorrelation function $A(s) = \frac{\langle x(t+s)x(t)\rangle}{\langle x^2(t)\rangle}$.



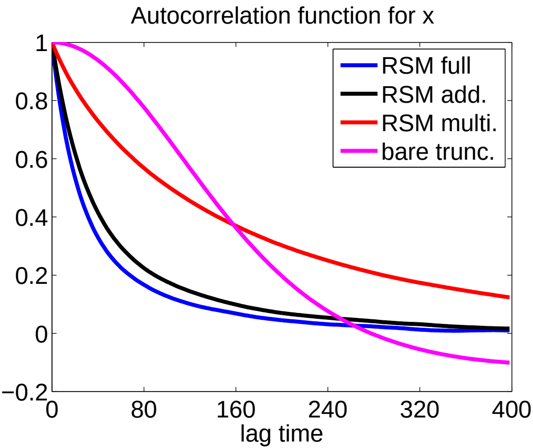
The reduced stochastic model

Grid point averaged kurtosis $K(s) = \frac{\langle x(t+s)^2 x^2(t) \rangle}{\langle x^2(t) \rangle^2 + 2\langle x(t+s)x(t) \rangle^2}$.



Outline

Inviscid case: different contributions in the RSM

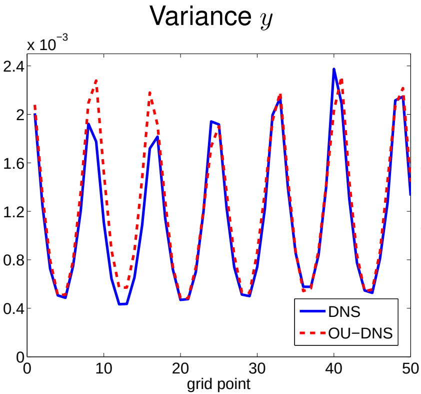


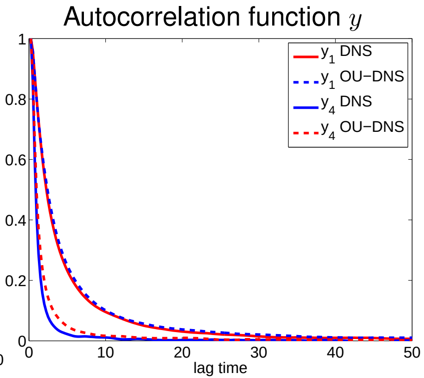
	RSM full	RSM add.	RSM multi.	bare trunc.
$Var(x)$	2.26	2.31	1.99	12.1
$\int \lvert ACF_s(x) \rvert ds$	47.3	61.0	149	144.7

Conclusions and outlook

- subgrid scale motion models for the forced and for the inviscid Burgers equation
- models derived applying a systematic stochastic mode reduction strategy
- generalization of the MTV strategy for an oscillating OU process
- additive and multiplicative noise in the effective equations
- compare this approach with “seamless MTV”
- subgrid scale closure for the barotropic vorticity equation

Test the OU-closure assumption





Averaged autocorrelation function from a simulations with 64 points

	$Var(y)$	$\int \lvert ACF_s(y) \rvert ds$	$Var(x)$	$\int \lvert ACF_s(x) \rvert ds$
DNS	$1.12\,10^{-3}$	27.9	$2.45\,10^{-2}$	147.6
OU-DNS	$1.21\,10^{-3}$	32.1	$2.46\,10^{-2}$	151.9

The reduced stochastic model for the inviscid case

The full model with OU-closure on the grid

$$dx_i(t) = \left(\sum_{j,k} B^{xxx}_{ijk} x_j x_k + \sum_{j,k} B^{xxy}_{ijk} x_j y_k + \sum_{j,k} B^{xyy}_{ijk} y_j y_k \right) dt \,,$$

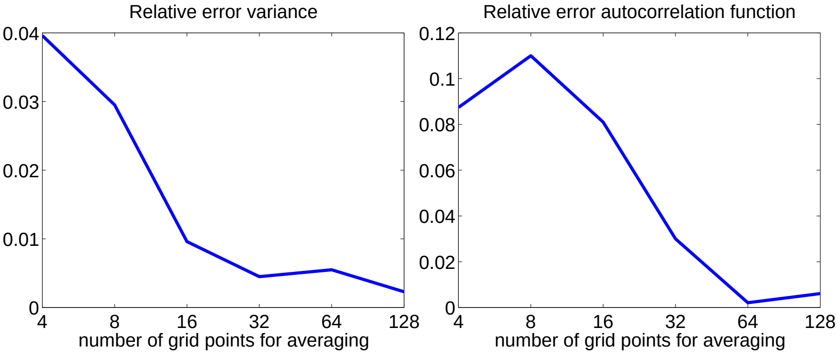
$$dy_i(t) = \left(\sum_{j,k} B^{yxx}_{ijk} x_j x_k + \sum_{j,k} B^{yxy}_{ijk} x_j y_k - \gamma y_i \right) dt + \sigma dW_i \,.$$

Applying the systematic MTV procedure, we obtain the following effective stochastic equation for $x_i(t)$

$$dx_i(t) = \left(\sum_{j,k=i-1}^{i+1} B^{xxx}_{ijk} x_j x_k + \sum_{j=i-1}^{i+1} \widehat{M}_{ij} x_j + \sum_{j,k,l=i-2}^{i+2} \widehat{C}_{ijkl} x_j x_k x_l \right) dt$$

$$+ \sum_{j,k \in \widehat{s}^{i+1}_{i-1}} \widehat{P}_{ijk} dW_{jk} + \sum_{j=i-1}^{i+1} \sum_{k \in \widehat{s}^{i+1}_{i-1}} \widehat{D}_{ijk} x_j dW_k \,.$$

Increasing the scale separation



Relative error for the RSM, when compared with the OU-DNS; variance (left) and autocorrelation function (right)

Different contributions in the RSM

The full model with OU-closure

$$dx_i(t) = \left(\sum_{j,k} B^{xxx}_{ijk} x_j x_k + \sum_{j,k} B^{xxy}_{ijk} x_j y_k + \sum_{j,k} B^{xyy}_{ijk} y_j y_k \right) dt \,,$$

$$dy_i(t) = \left(\sum_{j,k} B^{yxx}_{ijk} x_j x_k + \sum_{j,k} B^{yxy}_{ijk} x_j y_k - \gamma y_i \right) dt + \sigma dW_i \,.$$

The reduced stochastic model

$$dx_i(t) = \left(\sum_{j,k=i-1}^{i+1} B^{xxx}_{ijk} x_j x_k + \sum_{j=i-1}^{i+1} \widehat{M}_{ij} x_j + \sum_{j,k,l=i-2}^{i+2} \widehat{C}_{ijkl} x_j x_k x_l \right) dt$$

$$+ \sum_{j,k \in \widehat{s}^{i+1}_{i-1}} \widehat{P}_{ijk} dW_{jk} + \sum_{j=i-1}^{i+1} \sum_{k \in \widehat{s}^{i+1}_{i-1}} \widehat{D}_{ijk} x_j dW_k \,.$$