# Space-Time Regularizations for the ICON Model

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Berlin, 6 June 2011

# Outline

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# Motivation for space- and time-regularization

- Mathematical models for atmospheric and oceanic flows exhibit the following problems
  - time step in explicit schemes is restricted by the fastest modes (e.g., gravity, acoustic waves)
    Standard approach $\longrightarrow$ Semi-implicit time stepping schemes
  - processes on unresolved (subgrid) scales interact in complex ways with those on resolved scales
    Standard approach $\longrightarrow$ LES, RANS

- Our approach $\longrightarrow$ Introduce a regularized set of analytical equations which do not suffer from such problems

# Motivation for space- and time-regularization

Our work considers two types of regularization

- Time ($\tau$) regularization = Improve the computational efficiency by introducing a regularized (i.e. smoothed) pressure field
  - slows down the fastest waves
  - allows explicit time integration with a time step analogous to that of a semi-implicit method.
- Space (*alpha*) regularization = Parametrize the unresolved scales by introducing a regularized (i.e. smoothed) velocity field
  - conserves mass, potential vorticity, and potential enstrophy
  - preserves the validity of Kelvin's circulation theorem

Both regularized models

- can be interpreted as averaging out small scale fluctuations
- lead to analytic problems that can be proved to be well-posed
- do not enhance the viscosity of the flow

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# 3D Euler: The hydrostatic Euler equations

We consider a hydrostatic Euler system given by

$$\partial_t \mathbf{v} + (\mathbf{k} \cdot \nabla \times \mathbf{v} + f)\mathbf{v}^\perp + w\partial_z \mathbf{v} + c_p\, \theta\, \nabla\pi + \frac{1}{2}\nabla|\mathbf{v}|^2 = 0 \qquad (1)$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{v}) = 0 \qquad (2)$$

$$\frac{\partial\mu}{\partial t} + \nabla\cdot(\mu\,\mathbf{v}) = 0 \qquad (3)$$

augmented with a hydrostatic

$$c_p\, \theta\, \frac{\partial\pi}{\partial z} + g = 0 \qquad (4)$$

and a thermodynamic relation

$$\frac{R}{p_s}\rho\theta = \pi^{\frac{1-\kappa}{\kappa}}. \qquad (5)$$

*Notation:* $\mathbf{v} = (u, v)^T$ *– horizontal velocity,* $w$ *– vertical velocity,* $\rho$ *– density,* $\theta$ *– potential temperature,* $c_p$ *– specific heat of dry air,* $\mathbf{k} = (0, 0, 1)^T$*,* $g$ *– free-fall acceleration,* $f$ *– Coriolis parameter,* $p_s$ *– surface pressure,* $R$ *– gas constant,* $\pi = \mu^{\frac{\kappa}{1-\kappa}}$ *– Exner pressure,* $\kappa \approx 0.4$.

# 3D Euler: $\tau$-regularization

The $\tau$-regularized system is defined by

$$\partial_t \mathbf{v} + (\mathbf{k} \cdot \nabla \times \mathbf{v} + f)\mathbf{v}^{\perp} + w\partial_z \mathbf{v} + c_p\, \theta\, \nabla\tilde{\pi} + \nabla \frac{1}{2}|\mathbf{v}|^2 = 0 \quad (6)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \mathbf{v}) = 0 \quad (7)$$

$$\frac{\partial \mu}{\partial t} + \nabla \cdot (\mu\, \mathbf{v}) = 0, \quad (8)$$

where smoothed Exner pressure $\tilde{\pi}$ is computed from $\pi$ by solving

$$\left[1 - \tau^2 \frac{c_s^2}{\rho\theta^2} \nabla \cdot (\rho\theta^2 \nabla)\right] \tilde{\pi} = \pi + \tau^2 \mathcal{R} \quad (9)$$

with $\tau \geq 0$, a smoothing parameter, and $\mathcal{R}$ given by

$$\mathcal{R} = \frac{c_s^2}{c_p\, \rho\theta^2} \frac{\partial}{\partial z} \left( \frac{g\,\rho\theta}{1 + \tau^2 g \partial_z \log\theta} \right). \quad (10)$$

*Notation:* $c_s$ *– speed of sound*

# 3D Euler: $\alpha$-regularization

The $\alpha$-regularized system is defined by

$$\begin{aligned} \partial_t \mathbf{v} + (\mathbf{k} \cdot \nabla \times \mathbf{v} + f)\tilde{\mathbf{v}}^\perp + w\partial_z \mathbf{v} + c_p\, \theta\, \nabla\pi & & (11)\\ +\nabla \left[ \tilde{\mathbf{v}} \cdot \mathbf{v} - \frac{1}{2} \left( |\tilde{\mathbf{v}}|^2 + \alpha^2 |\nabla \tilde{\mathbf{v}}|^2 \right) \right] &= 0 & (12)\\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\, \tilde{\mathbf{v}}) &= 0 & (13)\\ \frac{\partial \mu}{\partial t} + \nabla \cdot (\mu\, \tilde{\mathbf{v}}) &= 0, & (14) \end{aligned}$$

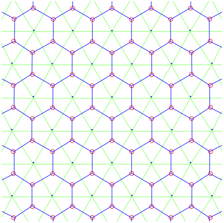
where $\tilde{\mathbf{v}}$ is computed from $\mathbf{v}$ by solving

$$\left[ 1 - \alpha^2 \nabla \cdot \nabla \right] \tilde{\mathbf{v}} = \mathbf{v}, \qquad (15)$$

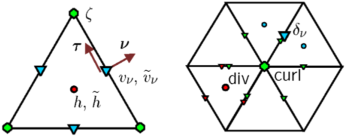
and $\alpha \geq 0$ is a smoothing parameter.

# 3D Euler: Horizontal discretization

The horizontal discretization follows the ICON framework (Bonaventura, 2005) and is based on a triangular grid

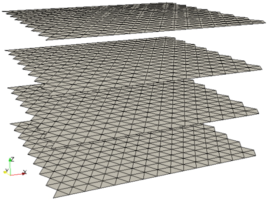


- The normal velocities in the horizontal are located at edge midpoints
- Pressure and transported variables are located at the cell centers



# 3D Euler: Vertical and time discretizations

- The vertical velocity $w$ is located on layer interfaces.
- All variables except $w$ are defined in the middle of a layer
- The vertical velocity is treated as a Lagrange multiplier that enforces the hydrostatic balance




- The regularization problem is solved in each horizontal layer; thus, we avoid solving a global 3D problem
- Several time stepping schemes are implemented
  - 3rd order total variation bounded (TVB) Runge-Kutta method (Cockburn et al, 1999)
  - 4th order strong stability preserving Runge-Kutta method (Spiteri, Ruuth, 2002)
  - Störmer-Verlet method

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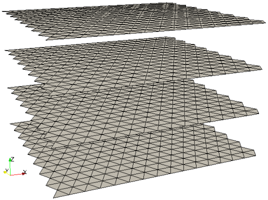

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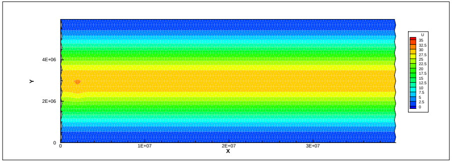
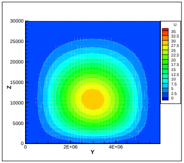
# Baroclinic instability – cartesian case

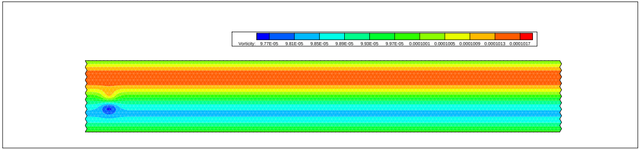
The 3D model is run for the Jablonowski baroclinic instability test on an f-plane (Jablonowski, 2006)

- Domain: $[0, 40000] \times [0, 6000] \times [0, 30]$ (in km)
- Horizontal resolution of $312.5\,km$
- 30 horizontal layers with $\Delta z = 1000\,m$
- The initial atmosphere is in hydrostatic and geostrophic balance except for a small wind perturbation
- Third order TVB Runge-Kutta method with $\Delta t$ varying from 600s (unregularized) up to 2400s ($\alpha$=1.25)




*Initial axial velocity; lengthwise at $z = 11000m$ (left), crosswise at x=20000km (right)*




*Initial vorticity at $z = 500m$*

# Restoration of the geostrophic balance

The geostrophic balance equation

$$f v^{\perp} + c_p \, \theta \, \nabla \pi = 0 \qquad (16)$$

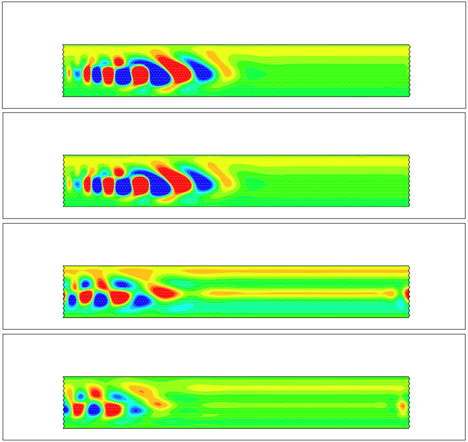
has for $\alpha$-regularized systems the form

$$f \tilde{\mathbf{v}}^{\perp} + c_p \, \theta \, \nabla \pi = 0 \qquad (17)$$

Possible solutions:

- Replace $\pi$ by $\tilde{\pi}$ that satisfies the geostrophic balance
  $\rightarrow$ Requires knowledge of the analytical form of $\tilde{\mathbf{v}}$, delays advance of the baroclinic instability due to smoother initial pressure field
- Assign the balanced velocity to $\tilde{\mathbf{v}}$ and define $\mathbf{v}$ using $\mathbf{v} = \left[1 - \alpha^2 \nabla \cdot \nabla\right] \tilde{\mathbf{v}}$
  $\rightarrow$ Can be computed automatically; produces a very rough initial velocity field for large $\alpha$, difficult to stabilize.

# Results(cartesian) – vorticity



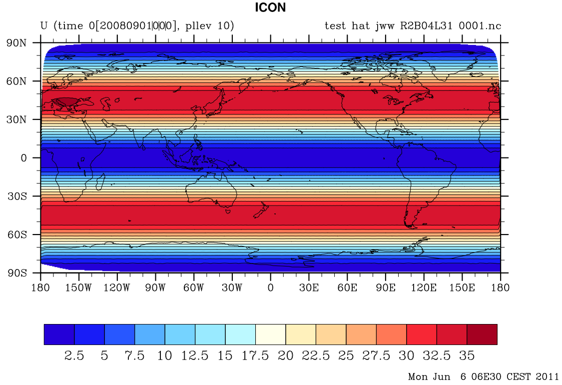
*Vorticity at $z = 500m$ at day 12. Top to bottom: no regularization, $\tau$-regularization, $\alpha$-regularization ($\alpha = 1$), $\alpha$-regularization ($\alpha = 1.25$)*

- The results of the $\tau$-regularization model are very similar to those of the unregularized system
- $\alpha$-regularization turns out to be very sensitive to the choice of alpha
- Initial conditions re-balanced by smoothing out the pressure field clearly slow down the advance of the instability

# Baroclinic instability – spherical case

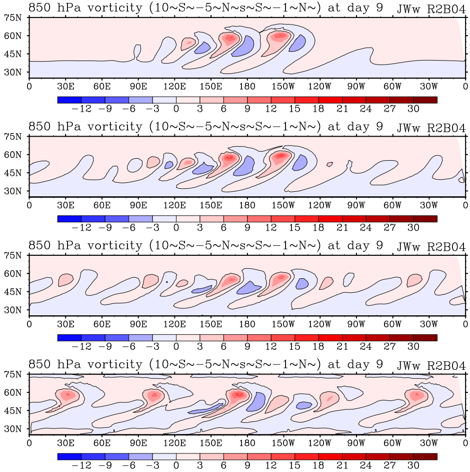
Jablonowski-Williamson baroclinic instability test on a sphere (Jablonowski, Williamson, 2007)

- Implementation within ICON hydrostatic core
- Initial condition re-balanced by appying the Helmholtz-operator to the horizontal velocity




*Initial axial velocity (m/s) at 850hPa*

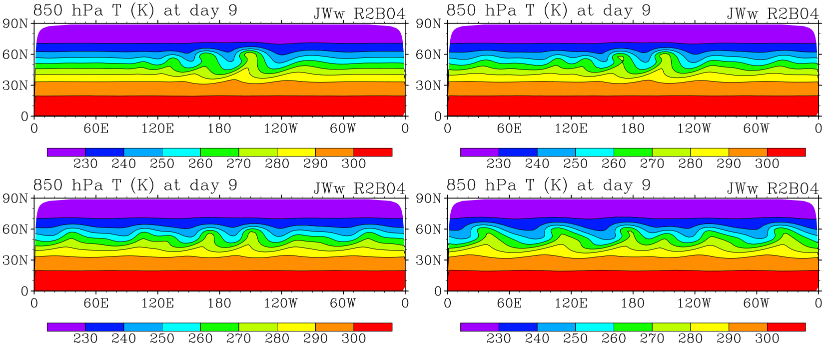
# Results(spherical) – vorticity




*850hPa vorticity at day 9. Top to bottom: no regularization, $\alpha$-regularization ($\alpha = 0.5$), $\alpha$-regularization ($\alpha = 0.8$), $\alpha$-regularization ($\alpha = 1$)*

- Re-balancing of the initial conditions by applying the Helmholtz-operator to the velocity field increases the instability
- $\alpha$-regularization allows for a much larger time steps than those of unregularized model

# Results(spherical) – temperature




*850hPa temperature at day 9. Top to bottom, left to right: no regularization, $\alpha$-regularization ($\alpha = 0.5$), $\alpha$-regularization ($\alpha = 0.8$), $\alpha$-regularization ($\alpha = 1$)*

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# Theoretical results

Theoretical investigation of time and space regularization models conducted in the framework of this project resulted in

- a systematic derivation of a pressure regularization for the vertical slice Euler model in combination with a Störmer-Verlet method (Hundertmark and Reich, 2007)
- proof of the dependence of the longtime behavior of solutions to the Navier-Stokes-$\alpha$, Leray-$\alpha$, and Navier-Stokes-$\omega$ systems on a finite set of grid values or Fourier modes (Korn, 2011)
- an estimate of the number of determining nodes/modes in terms of flow parameters for each model (as above)

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# Outlook and open questions

- *Boundary conditions:* The vast majority of theoretical and numerical work on the $\alpha$-equations considers periodic domains. The extension to bounded domains poses questions such as:
    - How to choose the mathematically reasonable & physically sound boundary conditions?
    - How the smoothing parameter should be adapted in the vicinity of a boundary?
- *Non-uniform grids:* Grids with varying spatial resolution might need a scale-selective filtering in the transition region between different resolutions.
    - How the smoothing parameter $\alpha$ should be chosen to provide an adequate filter?
    - What are the dispersion properties of regularized models on such grids?