Space-Time Adjustable Regularizations (STAR) for the Atmospheric Circulation Model ICON

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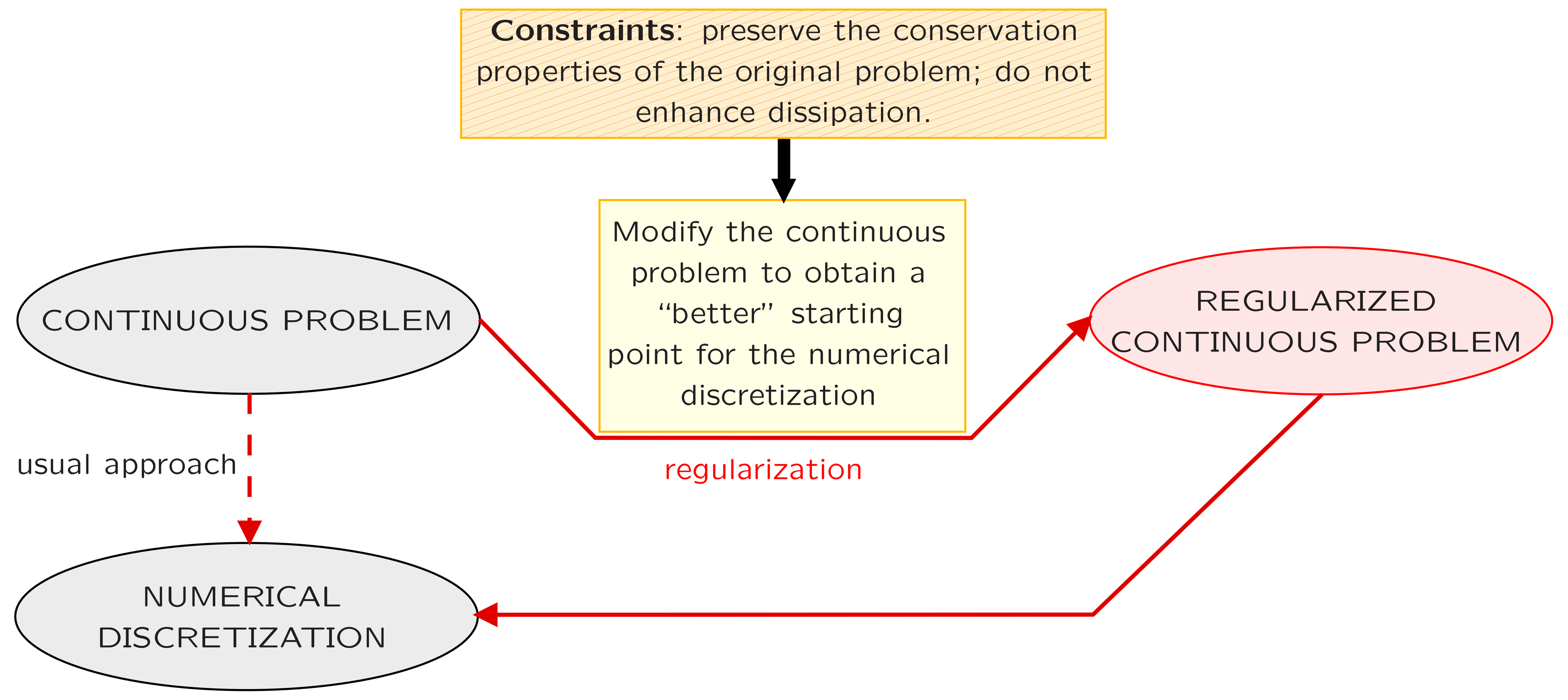
Overview



We are currently investigating two regularizations of the Navier–Stokes equations, both based on averaging out small scales fluctuations:

time regularization → improve the efficiency by enlarging the maximum stable time step;

space regularization → parametrize turbulence by reducing the energy cascade toward unresolved scales.

Notice that... the regularized problem turns out to be more tractable also from the analytic viewpoint, so that well posedness results can be obtained which are not known to hold for the original problem.

Regularized Continuous Problem

As a starting point, we consider here the shallow water (SW) equations. We denote by $\mathbf{v}$ the fluid velocity, by h the free surface elevation and by f and g the Coriolis parameter and the gravitational constant, respectively.

Time regularization

The time regularized SW system (SW-τ) is

$$\begin{aligned} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + f\mathbf{v}^{\perp} + g\nabla\widetilde{h} &= 0 \\ \partial_t h + \nabla\cdot[h\mathbf{v}] &= 0 \\ \left(1-\tau^2\nabla^2\right)\left[\widetilde{h}-h\right] &= \frac{\tau^2}{g}\nabla\cdot\mathcal{R} \end{aligned} \qquad (1)$$

where $\tau \geq 0$ is a smoothing parameter and $\widetilde{h}$ is the regularized layer depth.

- In analogy with a semi-implicit time integrator, SW-τ allows for large time steps by slowing down the fastest gravity waves.
- The choice $\mathcal{R} = g\nabla h$ in $(1)_3$ results in Hamiltonian equations and well-posedness of the problem [DLGP97].
- To avoid disturbing the geostrophic balance, one can take [RWS07]

  $$\mathcal{R} = g\nabla h + f\mathbf{v}^{\perp} + (\mathbf{v}\cdot\nabla)\mathbf{v}$$

  so that the regularization does not affect linearly/nonlinearly balanced flows.
- To maintain SW-τ as close as possible to the unfiltered problem, τ should be chosen as the smallest value still allowing for stable time integration, which leads to $\tau \propto \Delta t$, the constant of proportionality depending on the chosen time integrator.

Space regularization

The space regularized SW system (SW-α) is

$$\begin{aligned} \partial_t \mathbf{v} + \widetilde{\mathbf{v}}\cdot\nabla\mathbf{v} + \textstyle\sum_{j=1}^{2} v_j\nabla\widetilde{v}_j + f\widetilde{\mathbf{v}}^{\perp} & \\ + \nabla\left[gh - \tfrac{1}{2}\left(|\widetilde{\mathbf{v}}|^2 + \alpha^2|\nabla\widetilde{\mathbf{v}}|^2\right)\right] &= 0 \\ \partial_t h + \nabla\left[h\widetilde{\mathbf{v}}\right] &= 0 \\ \left(h - \alpha^2\nabla\cdot(h\nabla)\right)\widetilde{\mathbf{v}} &= h\mathbf{v}, \end{aligned} \qquad (2)$$

where $\widetilde{\mathbf{v}}$ is the regularized velocity and $\alpha \geq 0$ is the (uniform) smoothing parameter, usually set as $\alpha \approx \Delta x$.

System (2) has the following characteristics:

- it can be derived within an Hamiltonian framework [Hol99] by
  1. splitting the flow trajectories into mean and fluctuating components,
  2. averaging the Lagrangian functional over the fluctuating component,
  3. applying Hamilton's variational principle to the averaged Lagrangian functional;
- yields conservation of energy and enstrophy;
- reduces the energy/enstrophy cascade toward small scales [NS01];
- it has been found to give better results compared to the unfiltered SW system for ocean modeling [HHPW08].

Numerical Discretization

The spatial discretization follows [BR05] and is based on a triangular C-grid where the degrees of freedom are staggered according to Fig. 1, left.

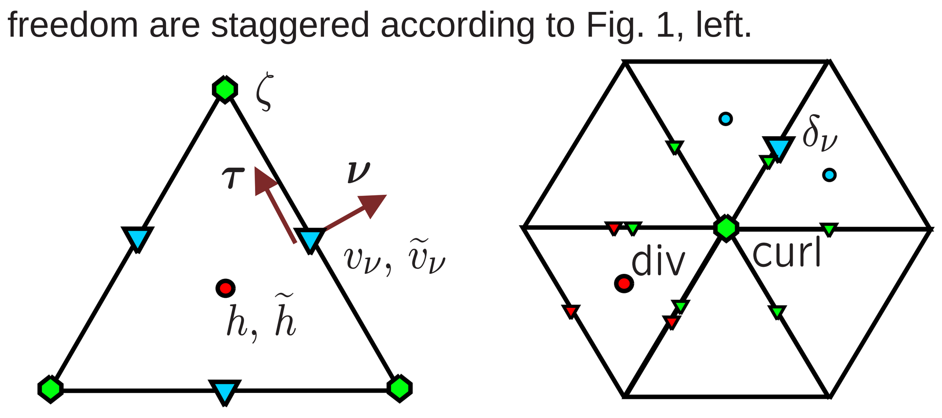


Figure 1. *C-grid staggering, left, and stencils of the discrete operators $\delta_{\vec{\nu}}$ (blue), curl (green) and div (red), right.*

The equations are solved in the invariant form with prognostic variables h and $v_{\vec{\nu}}$, with $v_{\vec{\nu}} = \mathbf{v}\cdot\boldsymbol{\nu}$

$$\begin{aligned} \partial_t v_{\vec{\nu}} &= \left(\mathrm{curl}\,(v_{\vec{\nu}}) + f\right) v^*_{\vec{\tau}} - \delta_{\vec{\nu}}\left[gh^* + p^*_D\right] \\ \partial_t h &= -\mathrm{div}\left(h v^*_{\vec{\nu}}\right), \end{aligned} \qquad (3)$$

with $h^* = \widetilde{h}$, $\mathbf{v}^* = \mathbf{v}$ and $p^*_D = \frac{|\mathbf{v}|^2}{2}$ for SW-τ and $h^* = h$, $\mathbf{v}^* = \widetilde{\mathbf{v}}$ and $p^*_D = \widetilde{\mathbf{v}}\cdot\mathbf{v} - \frac{1}{2}\left(|\widetilde{\mathbf{v}}|^2 + \alpha^2|\nabla\widetilde{\mathbf{v}}|^2\right)$ for SW-α. The operators curl, div and $\delta_{\vec{\nu}}$ use the stencils of Fig. 1, right. The regularized quantities $\widetilde{h}$, $\widetilde{\mathbf{v}}$ are diagnosed solving the elliptic problems $(1)_3$ and $(2)_3$, respectively.

The discretization (3) conserves mass, vorticity and enstrophy for both SW-τ and SW-α.

For the time discretization we use the 4$^{\mathrm{th}}$-order RK method described in [SR02].

Project Status

- The numerical approach of [BR05] has been consistently extended to SW-τ and SW-α.
- SW-τ and SW-α have been implemented in the ICON GCM model.
- The numerical results confirm the efficiency of SW-τ and show the effect of SW-α on energy spectra and delay of flow instability.

References

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Numerical Experiments

Shear flow instability

Simulation of the instability of an inviscid shear flow. Results for SW-τ, SW-α and a reference unfiltered SW computation.

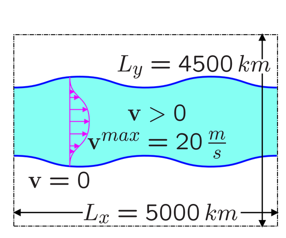
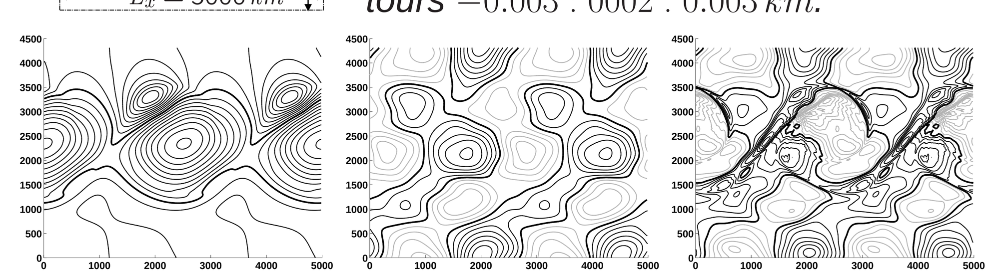
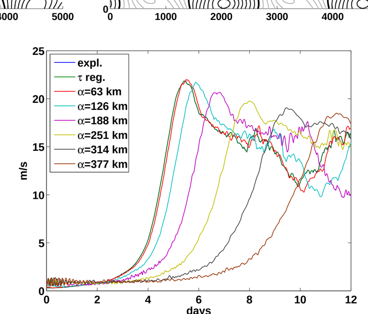


Figure 2. *Initial condition (left), middle: h_{sw} (left), contours $1.016 : 0.004 : 1.116\,km$, $h_{sw} - h_{sw\text{-}\tau}$ (center) and $h_{sw} - h_{sw\text{-}\alpha}$ (right), both with contours $-0.003 : 0002 : 0.003\,km$.*



Time evolution of the maximum meridional velocity in SW-α for different values of α (right). Larger $\alpha \Rightarrow$ delayed instability.



Rossby–Haurwitz wave

Test case 6 of [WDH+92], spherical geometry.

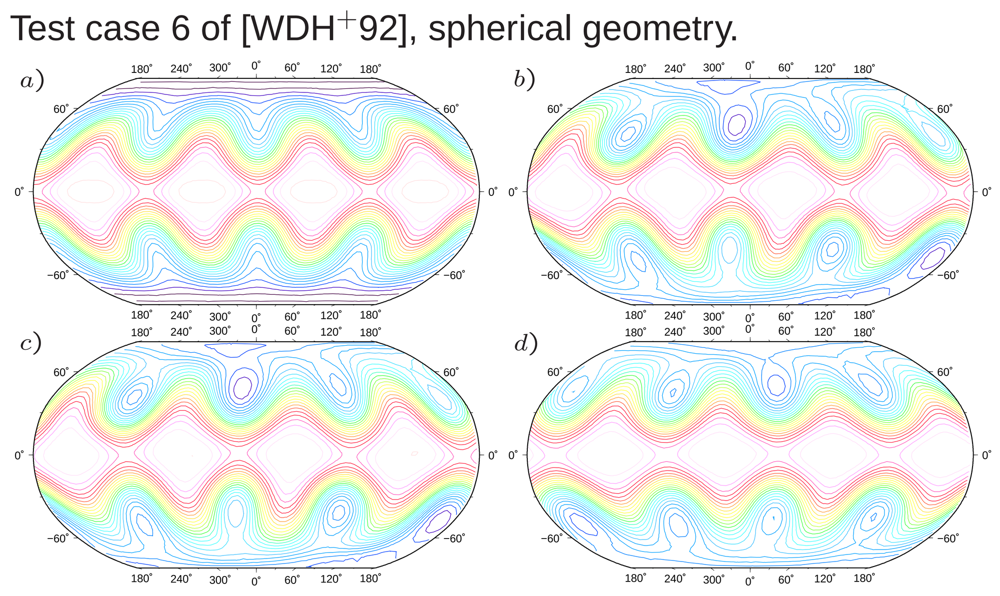


Figure 3. *Free surface height after $12\,days$, contours from $8030\,m$ to $10530\,m$, contour intervals of $100\,m$. Analytic solution (a), reference explicit solution with $\Delta t = 900\,s$ (b), time and space regularized solutions (c and d, respectively), both with $\Delta t = 2400\,s$.*

Decaying turbulence

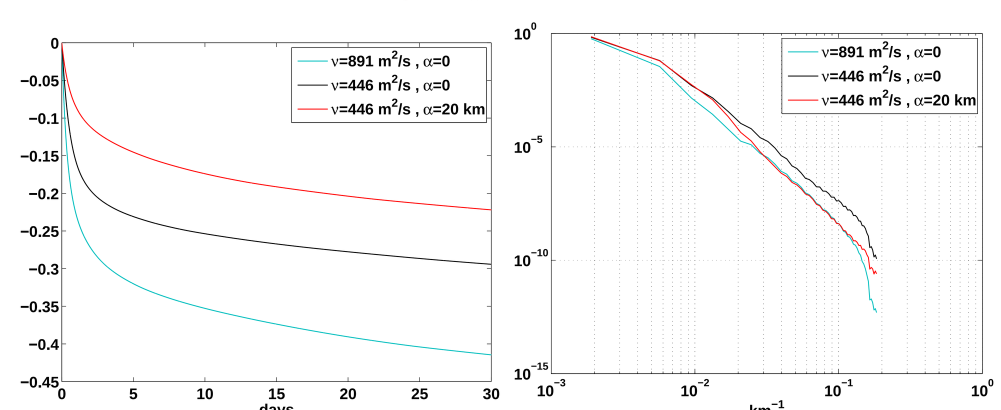


Figure 4. *Decaying turbulence experiment: SW with viscosity $\nu = 891\,m^2s^{-1}$ (blue), and $\nu = 446\,m^2s^{-1}$ (black), and SW-α with $\nu = 446\,m^2s^{-1}$ and $\alpha = 20\,km$ (red). Left: energy dissipation; right: energy spectra at day 30. Results for SW-τ, not plotted, analogous to those of SW.*

Notice that:

- the space regularization does not enhance dissipation;
- the space regularization enhance the backward energy cascade, a result analogous to [NS01].