Turbulence Statistics
along Gradient Trajectories

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Introduction

Turbulence: phenomenologically a fluid regime characterized by chaotic and stochastic property changes.

by Leonardo da Vinci
Various applications

combustors
aircrafts
meteorology
Methods used in turbulence research

• Experiments
  • advantages: real physical phenomena; high Reynolds numbers
  • disadvantages: only limited access to full 3D structures, often straight line measurements

• Direct numerical simulation (DNS)
  • advantages: high spatial resolution, entire velocity and pressure fields
  • disadvantages: artificial effects from boundary conditions, low Reynolds numbers.

• Stochastic theory and scaling
Kolmogorov’s (1941) first hypothesis of similarity

For locally isotropic turbulence

the n-point distribution functions $F_n$

are uniquely determined by the viscosity $\nu$ and the dissipation $\varepsilon$

Inertial range: $\nu \to 0$, $\varepsilon$–scaling!
Two point statistics along a straight line

structure function of moment $m$: $B_m = \langle \left( u'(x + r, t) - u'(x, t) \right)^m \rangle$

Kolmogorov’s equation

$$3 \frac{\partial B_2}{\partial t} + \frac{1}{r^4} \frac{\partial}{\partial r} \left( r^4 B_3 \right) = -4 \varepsilon + \frac{6\nu}{r^4} \frac{\partial}{\partial r} \left( r^4 \frac{\partial B_2}{\partial r} \right)$$

$= 0$ for steady state

exact result: $B_3 = -\frac{4}{5} \varepsilon r$

scale invariance assumption: $B_m \propto (\varepsilon r)^{\zeta_m}$

with $\zeta_m = \frac{m}{3}$ for all moments
If Kolmogorov’s scale invariance was exact, the task of computing practical flows would be relatively simple.

The eddy viscosity relating the third to the second structure function would then be

$$\nu_t = \alpha \cdot r \sqrt{B_2(r)}$$

where $\alpha$ is a universal constant (Oberlack & Peters, 1993).

**Scale invariance** would then provide a general framework for developing closure models.
Unfortunately, scale invariance is *not* exact

Examples:

- Anomalous scaling - scaling exponents $\zeta_m$ depart from $m/3$ for $m \neq 3$

- Derivative skewness and flatness are Reynolds number dependent

Why should we care?

We have two-equations models of turbulence to close the Reynolds averaged Navier Stokes equations (RANS),

even better: we have Large Eddy Simulations

But:

*The basic argument in favor of modeling unclosed expressions is scale invariance for the unresolved scales*
Conditional scaling along gradient trajectories in a scalar field

A scalar field can be that of a passive scalar, the instantaneous kinetic energy or the instantaneous dissipation
Chaotic motion of gradient trajectories in a 2-D scalar field
Cliff-ramp structure in the scalar field

(distribution along a horizontal line)
Two point statistics along gradient trajectories

Scalar gradient:

\[ \phi', n = \frac{\partial \phi'}{\partial n} \]

Two point correlation:

\[
C_2 = \langle \phi', n_{(n+s,t)} \phi', n_{(n,t)} \rangle
\]

\[
\frac{\partial C_2}{\partial t} + \frac{\partial}{\partial s} \left( \langle u'_{(n+s,t)} - u'_{(n,s)} \rangle \phi', n_{(n+s,t)} \phi', n_{(n,t)} \right) = D(...)
\]

\[ \propto - \frac{C_2}{\tau} \]

\[ \frac{\langle u'_{(n+s,t)} - u'_{(n,s)} \rangle C_2}{s} = 0 \text{ for } \nu \to 0 \]

for \( t \to \infty \)

Due to decorrelation

Result:

\[ \langle u'_{(n+s,t)} - u'_{(n,s)} \rangle = c_1 \frac{s}{\tau} \text{ for } s > \lambda \]

scaling parameter is the \textbf{integral time} \( \tau \) rather than \( \varepsilon \).
Normalized velocity increments along gradient trajectories in the passive scalar field in shear flow turbulence

\[ \Delta u \cdot (\tau / \lambda) \]

Linear scaling \[ \Delta u = c_1 \frac{s}{\tau} \]
Normalized velocity increments along gradient trajectories in the kinetic energy field for different flow configurations

![Graph showing normalized velocity increments along gradient trajectories for different flow configurations](image)

<table>
<thead>
<tr>
<th>flow type</th>
<th>grid</th>
<th>Re_τ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Homogeneous shear turbulence</td>
<td>$2048^3$</td>
<td>295</td>
</tr>
<tr>
<td>Homogeneous shear turbulence</td>
<td>$1024^3$</td>
<td>139</td>
</tr>
<tr>
<td>Isotropic homogeneous forced turbulence</td>
<td>$1024^3$</td>
<td>126</td>
</tr>
<tr>
<td>Kolmogorov flow</td>
<td>$1024^3$</td>
<td>188</td>
</tr>
<tr>
<td>Isotropic homogeneous decaying turbulence</td>
<td>$1024^3$</td>
<td>71</td>
</tr>
</tbody>
</table>

Linear scaling: $\Delta u \sim \frac{s}{\tau}$

but with different slopes
## Two different scalings

<table>
<thead>
<tr>
<th></th>
<th><strong>ε-scaling</strong></th>
<th><strong>τ-scaling</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity increment at large scales</td>
<td>( \delta_r u \propto (\varepsilon l)^{1/3} )</td>
<td>( \delta_r u \propto l/\tau )</td>
</tr>
<tr>
<td>velocity decay at small scales</td>
<td>( \delta_r u \propto \nu/l )</td>
<td>( \delta_r u \propto \nu/l )</td>
</tr>
<tr>
<td>transition at equal ( \delta_r u )</td>
<td>( (\varepsilon l_c)^3 = \nu/l_c )</td>
<td>( l_c/\tau = \nu/l_c )</td>
</tr>
<tr>
<td>critical cut-off scale</td>
<td>( l_c = \eta = \left( \frac{\nu^3}{\varepsilon} \right)^{1/4} )</td>
<td>( l_c = (\nu \tau)^{1/2} = \lambda )</td>
</tr>
</tbody>
</table>

**Kolmogorov scale** \( l_c \)  
**Taylor scale** \( \lambda \)
Dissipation elements

Local minimum and maximum points in the mixture fraction fluctuation field are determined by gradient trajectories starting from each grid cell in the directions of ascending and descending scalar gradients.

Definition:

The ensemble of grid cells from which the same pair of extremal points is reached determines a spatial region defined as “dissipation element“.
Interaction of dissipation elements with vortex tubes
Parametric description

Among the many parameters to describe the statistical properties of dissipation elements, we have chosen $l$ and $\Delta \phi'$, which are defined as the straight line connecting the two extremal points and the scalar difference at these points, respectively.
Extremal points and strain rates for the scalar field in homogeneous shear flow

Clustering of extremal points becomes more evident.

Experimental setup in the wind tunnel of the Aerodynamics Institute at the RWTH Aachen (Prof. Schröder)
Tomographic PIV and visualisation of dissipation elements

(L. Schäfer, Physics of Fluids 23 (2011), 035106)
Experimental investigation of the mass fraction field of propane discharging from a round jet into ambient air (I)

Nd:YLF* Laser
- diode pumped double cavity

Optics
- Cam: up to 10kHz and 1024² pixel

Combination of Taylor’s hypothesis and high-speed Rayleigh scattering

*neodymium-doped yttrium lithium fluoride
Experimental investigation of the mass fraction field of propane discharging from a round jet into ambient air (II)
Experimental investigation of the mass fraction field of propane discharging from a round jet into ambient air (III)

Maximum (red) and minimum (blue) of dissipation elements and their linear connecting line (green)
Joint pdf of scalar difference at the extremal points and the linear length from DNS calculations

Conditional mean follows the Kolmogorov scaling.

Experimental data using Rayleigh scattering for the joint pdf of element length and scalar difference
The marginal pdf of length for the passive scalar field from DNS

- The normalized shape of the pdf is Reynolds number independent.
Experimental and DNS data for marginal pdf’s in turbulent channel flow

- Experimental data are from 50 test volumes
- Good agreement of pdf’s regarding exponential decay
- Average dissipation element length scales with Taylor length

(L.Schäfer, Physics of Fluids 23 (2011), 035106)
Experimental and DNS data for the scalar field

• Excellent agreement between experimental data and model solution for marginal pdf at $x/d=30$

• Linear increase at the origin due to diffusion

• Exponential tail modeled by Poisson process

• Very good agreement of maximum position and value
A model for the length pdf

Rapid (jump) processes:

1. The Poisson processes of random splitting and (re-) attachment.

   This gives an exponential distribution for large elements.

Slow processes

2. Continuous change of length by diffusion and straining of end points.

   Diffusive drift to origin enforces the $P(l=0)=0$. 
Evolution equation for the linear length

There are four terms describing the changes of the pdf

- **Generation (of small elements) by splitting**
- **Removal (of all elements) by attachment**
- **Generation and Removal (of different size elements) by strain**
- **Removal (of small elements) by diffusional drift**

\[
\frac{\partial P(l,t)}{\partial t} + \frac{\partial [D/lP(l,t)]}{\partial l} + \frac{\partial [a(l)lP(l,t)]}{\partial l} = \lambda_s \int_l^\infty yP(y,t)dy - \mu_a lP(l,t)
\]

- **diffusional drift**
- **drift due to strain**
- **splitting**
- **attachment**

Parameters \( D = \nu \) and strain \( a(l) \sim 1/\tau \) leads to the **Taylor scale**.
Conditional mean strain rate of dissipation elements

\[ \Delta u = a(l) \cdot l \]

Scaling with \( a_\infty \) instead of \( 1/\tau \) collapses the normalized velocity increments of different flow configurations.
Derivation of the $e$-equation by taking appropriate moments of the evolution equation

For homogeneous shear turbulence

$$\frac{\partial \varepsilon}{\partial t} = c_{\varepsilon_1} (-u'u') \frac{\varepsilon}{k} \frac{\partial \bar{u}}{\partial y} - c_{\varepsilon_2} \frac{\varepsilon^2}{k}$$

Standard values: $c_{\varepsilon_1} = 1.44$, $c_{\varepsilon_2} = 1.9$

Are the constants (!) $c_{\varepsilon_1}$ and $c_{\varepsilon_2} = 1.9$ Reynolds-number dependent?

Consider decaying turbulence: $k \sim (t - t_0)^{-m}$, $m = \frac{1}{c_{\varepsilon_2} - 1}$

Experimental data: $m = 1.25 \rightarrow c_{\varepsilon_2} = 1.8$

Final stage, Re $\rightarrow 0$: $m = 1.5 - 2.5 \rightarrow c_{\varepsilon_2} = 1.66 - 1.4$
Scalar fields of kinetic energy \( k \) and dissipation \( \varepsilon \)

- **kinetic energy \( k \)**
- **dissipation \( \varepsilon \)**
Starting point:

Conditional dissipation

\[ \langle \varepsilon | l \rangle = \varepsilon^* \left( \frac{l}{l_m} \right)^n \]

Exponent \( n \) is Reynolds-number dependent
Relation to $\varepsilon$-equation

$$\frac{\partial \varepsilon}{\partial t} = a_\infty \varepsilon^* \left( I_s - I_a - I_{\text{strain}} - I_{\text{drift}} \right)$$

\[\text{production} \quad \text{dissipation}\]

**Production term:**
\[a_\infty \varepsilon^* I_{\text{prod}} \sim \varepsilon^* \frac{\partial \bar{u}}{\partial y} \sim c_{\varepsilon 1}(\text{Re}) \left( -u'v' \right) \frac{\varepsilon \partial \bar{u}}{k \partial y}\]

If mean length $l_m$ is proportional to the Taylor scale:
\[l_m^2 \sim \chi^2 = 10 \nu \frac{k}{\varepsilon}\]

**Dissipation term:**
\[a_\infty \varepsilon^* I_{\text{drift}} \sim \varepsilon \nu / l_m^2 \sim c_{\varepsilon 2}(\text{Re}) \frac{\varepsilon^2}{k}\]

Since $n$ is Reynolds number dependent, so must be $c_{\varepsilon 1}$ and $c_{\varepsilon 2}$. 

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<table>
<thead>
<tr>
<th></th>
<th>case 1</th>
<th>case 2</th>
<th>case 3</th>
<th>Empirical value</th>
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<tbody>
<tr>
<td>( \text{Re}_\lambda )</td>
<td>98.7</td>
<td>125.0</td>
<td>170.0</td>
<td>—</td>
</tr>
<tr>
<td>( c_{\varepsilon , 1} )</td>
<td>0.425</td>
<td>0.763</td>
<td>1.20</td>
<td>1.44</td>
</tr>
<tr>
<td>( c_{\varepsilon , 2} )</td>
<td>0.457</td>
<td>0.923</td>
<td>1.64</td>
<td>1.90</td>
</tr>
</tbody>
</table>
Conclusions

1. While Kolmogorov’s $\varepsilon$-scaling laws tell us how much energy is contained in an element of size $l$, the pdf of linear length provides the additional information on how many elements of size $l$ are contained in the flow.

2. This pdf equation contains diffusive effects with $D=\nu$ and a $\tau$-scaling due to strain and leads to the Taylor scale as mean length scale of dissipation elements.

3. Using dissipation elements to reconstruct the $\varepsilon$-equation reproduces the form of the equation but shows a Reynolds number dependence of the empirical modeling constants.
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Thank you for your attention
Two point correlation of the scalar gradient

along a straight line

- correlation becomes small for large $l$
- scalar gradient decorrelates from velocity difference

along a gradient trajectory
Results: anomalous scaling exponents

- DE analysis from DNS
- joint PDF model from DE analysis
- Experiment, Antonia et al.
Kolmogorov flow